e and π are transcendental

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The proofs below follow [1], [2] and [3].

Justification plus progressive des éléments dans la vidéo :
https://www.youtube.com/watch?v=WyoH_vgiqXM

1 e is transcendental

We start by assuming that e is the root of a non-zero integer-coefficient polynomial. Choosing such a polynomial of lowest degree, we would then have

$$a_n e^n + a_{n-1}e^{n-1} + \cdots + a_0 = 0\,, \qquad a_0 \neq 0. \tag{1}$$

In order to prove that this is impossible, we'll show that we can write

$$e^k = \frac{N_k + \delta_k}{N}\,, \qquad k = 1, \cdots, n, \tag{2}$$

with N and the N_k integers, and the δ_k tiny. Substituting into (1) and multiplying by N, we will then have

$$a_0 N + (a_1 N_1 + \cdots + a_n N_n) + (a_1\delta_1 + \cdots + a_n\delta_n) = 0\,. \tag{3}$$

We shall construct the approximations so that the integer part of this expression is non-zero and the δ part has magnitude less than 1. That will then be a contradiction, proving (1) is impossible.

The proof uses an integral relation between the function e^{-x} and factorials. Specifically, with p a large prime to be determined later, we have[1]

$$\boxed{\frac{1}{(p-1)!}\int_0^\infty e^{-x}x^j\,\mathrm{d}x = \frac{j!}{(p-1)!} = \begin{cases} 1, & j = p-1, \\ \text{a multiple of } p, & j \geqslant p. \end{cases}} \tag{4}$$

Note also that if f is any polynomial then, as long as the denominator is not zero, trivially

$$e^k = \frac{\int_0^\infty e^{k-x}f(x)\,\mathrm{d}x}{\int_0^\infty e^{-x}f(x)\,\mathrm{d}x}\,. \tag{5}$$

We now choose (justifié dans la vidéo)

$$f(x) = x^{p-1}(x-1)^p(x-2)^p\cdots(x-n)^p\,. \tag{6}$$

[1] This is a standard integration by parts exercise. See, for example, the Wikipedia page on the gamma function.

For $k = 1, \cdots, n$ we then define

$$\boxed{\begin{cases} N = \dfrac{1}{(p-1)!} \displaystyle\int_0^\infty e^{-x} f(x)\,\mathrm{d}x\,, \\[2ex] N_k = \dfrac{1}{(p-1)!} \displaystyle\int_k^\infty e^{k-x} f(x)\,\mathrm{d}x\,, \\[2ex] \delta_k = \dfrac{1}{(p-1)!} \displaystyle\int_0^k e^{k-x} f(x)\,\mathrm{d}x\,. \end{cases}}$$

Noting that

$$f(x) = (-1)^p(-2)^p \cdots (-n)^p x^{p-1} + \text{higher powers of } x,$$

it follows from (4) that

$$N = (-1)^{np}\,(n!)^p + \text{a multiple of } p.$$

So, N is an integer. Moreover, if we choose the prime p to be larger than n, then $n!$ cannot be a multiple of p, and so neither is N. In particular, $N \neq 0$ and (2) is now immediate from (5). If, further, we ensure $p > |a_0|$ then, noting $a_0 \neq 0$, it follows that $a_0 N$ also cannot be a multiple of p.

Next, we perform the substitution $t = x - k$ in the integral for N_k, giving

$$N_k = \frac{1}{(p-1)!} \int_0^\infty e^{-t} f(t+k)\,\mathrm{d}t\,.$$

Clearly $f(t+k)$ has a factor t^p, and so from (4) again, N_k *is* an integral multiple of p. It follows that the N part of (3) is a non-zero integer, as desired.

It remains to show that if p is large then δ_k is tiny, and this is just the standard business. We just have to note that $|x-k| \leqslant n$ on $[0,n]$, and so on this interval

$$|f| \leqslant n^{(np+p-1)}\,.$$

Applying this estimate, we see

$$\delta_k \leqslant \frac{c \cdot d^p}{(p-1)!}\,,$$

with $c = e^n$ and $d = n^{(n+1)}$. It follows that $\delta_k \to 0$ as $p \to \infty$, and we're done.

2 π is transcendental

By way of contradiction, we assume that π is the root of a non-zero polynomial p with integer coefficients,

$$p(\pi) = 0\,.$$

This would imply that $i\pi$ is a root of the integer polynomial

$$q(x) = p(ix)p(-ix)\,.$$

We show that this latter equation leads to a contradiction. We first give the main argument, and in §2.2 we fill in some details on the symmetric polynomials employed in the argument.

2.1 The Main Argument

Write

$$q(x) = a(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)\,, \qquad a \neq 0. \tag{7}$$

with $\alpha_1 = i\pi$ and $a \in \mathbb{Z}$. Then $1 + e^{\alpha_1} = 0$, and so trivially

$$(1+e^{\alpha_1})\,(1+e^{\alpha_2})\cdots(1+e^{\alpha_n}) = 0\,. \tag{8}$$

Expanding gives

$$e^{\beta_1} + e^{\beta_2} + \cdots + e^{\beta_{2^n}} = 0\,, \tag{9}$$

where β_k ranges over all sums of distinct α_j, including the empty sum. Letting $\beta_1, \beta_2, \cdots, \beta_m$ be the *non-zero* β, we then have the key identity

$$r + e^{\beta_1} + e^{\beta_2} + \cdots + e^{\beta_m} = 0\,, \tag{10}$$

where $r = 2^n - m$. Note that $r > 0$, since at least the empty sum of α gives $\beta = 0$.

mais cette fois avec des β complexes

We can now mimic and adapt the proof that e is transcendental. To this end, for z complex define

$$f(z) = z^{p-1}g^p\,, \tag{11}$$

where p is a large prime to be determined later, and

$$g(z) = a^m(z-\beta_1)(z-\beta_2)\cdots(z-\beta_m)\,. \tag{12}$$

We now define the N and δ quantities analogously to the definitions in the e proof, but in terms of complex line integrals:[2]

$$\begin{cases} N = \dfrac{1}{(p-1)!}\displaystyle\int_0^\infty e^{-z} f(z)\,\mathrm{d}z\,, & \text{(along the positive real axis)}, \\[2ex] N_k = \dfrac{1}{(p-1)!}\displaystyle\int_{\beta_k}^\infty e^{\beta_k - z} f(z)\,\mathrm{d}z\,, & \text{(along the horizontal line from } \beta_k \text{ to } +\infty), \\[2ex] \delta_k = \dfrac{1}{(p-1)!}\displaystyle\int_0^{\beta_k} e^{\beta_k - z} f(z)\,\mathrm{d}z\,, & \text{(along the radial path from 0 to } \beta_k). \end{cases}$$

[2] So, for example, the radial path from 0 to β_k can be parametrised by $z = \beta_k t$ with $0 \leqslant t \leqslant 1$. Then $\mathrm{d}z = \frac{\mathrm{d}z}{\mathrm{d}t}\mathrm{d}t = \beta_k \mathrm{d}t$, and so on. The subsequent integral is complex-valued but it can be interpreted, computed and estimated with standard real-variable techniques.

Then, by a routine application of Cauchy's integral theorem,[3]

$$N_k + \delta_k = N e^{\beta_k} \,. \tag{13}$$

So, once we determine $N \neq 0$, it will follow from (10) that

$$rN + (N_1 + \cdots + N_m) + (\delta_1 + \cdots + \delta_m) = 0 \,. \tag{14}$$

Now, q has integer coefficients, and g is symmetric in $\beta_1, \cdots, \beta_m$, with a suitably large power of a as a factor. It then readily follows that g, and so also f, has integer coefficients; see §2.2, below. We can now choose the prime p to be larger than the magnitude of the constant term of g. Then, as in the proof for e, (4) implies that N is an integer and is not divisible by p, and in particular $N \neq 0$. If we further choose p larger than r, then rN also will not be divisible by p.

Next, setting $M = \max_k |\beta_k|$, it is easy to see that if $|z| \leqslant M$ then

$$|f(z)| \leqslant |a|^{mp} \, (2M)^{mp} \, M^{p-1} \,.$$

It follows that

$$\delta_k \leqslant \frac{c \cdot d^p}{(p-1)!} \,,$$

where $c = e^M$ and $d = 2^m |a|^m M^{m+1}$. So, $\delta_k \to 0$ as $p \to \infty$.

It remains to consider the N_k, which will in general be complex. We can complete the contradiction, however, by showing that the *sum* $N_1 + \cdots + N_m$ is an integral multiple of p, implying that the integral part of (14) is non-zero. In order to do this, we make the substitution $w = z - \beta_k$ in the integral for N_k. Summing then gives

$$\boxed{N_1 + \cdots + N_m = \frac{1}{(p-1)!} \int_0^\infty e^{-w} h(w) \, \mathrm{d}w \,,}$$

with the integral along the positive real axis, and where

$$h(w) = f(w + \beta_1) + \cdots + f(w + \beta_m) \,. \tag{15}$$

It is clear from the form of f that $h(w)$ has a factor w^p. As well, using that q has integer coefficients and that h is symmetric in $\beta_1, \cdots, \beta_m$, it is straight-forward to show that h has integer coefficients; see §2.2. It then follows from (4) that $N_1 + \cdots + N_m$ is an integral multiple of p, and we have our contradiction.

[3]Let T be a large real number and let γ be the closed parallelogram path through the vertices 0, β_k, $\beta_k + T$ and T. Cauchy's theorem says $\int_\gamma F = 0$ for our (complex differentiable) function F; see Wikipedia or, for example, the nice presentation at `people.reed.edu/~jerry/311/cauchy.pdf`. Now let $T \to +\infty$. Noting that $e^{-T} \to 0$ rapidly, it is easy to prove that $\int_T^{\beta_k + T} F \to 0$, and (13) follows.

2.2 Symmetric Polynomials

We need to show that f given by (11) and (12), and h given by (15), have integer coefficients. The arguments are standard applications of the fundamental theorem of symmetric polynomials: *any integer-coefficient symmetric polynomial is an integer-coefficient polynomial function of the elementary symmetric polynomials.*[4]

The theorem is of use to us because the integer coefficients of q are $\pm a$ times the elementary symmetric polynomials of $\alpha_1, \cdots, \alpha_n$. It follows that any symmetric integer polynomial of $a\alpha_1, \cdots, a\alpha_n$ is an integer, and thus the same is also true for any symmetric integer polynomial of $a\beta_1, \cdots, a\beta_{2^n}$.

We can now apply this to the polynomial

$$G(z) = (z - a\beta_1)(z - a\beta_2) \cdots (z - a\beta_m) = \frac{1}{z^r}(z - a\beta_1)(z - a\beta_2) \cdots (z - a\beta_{2^n}) . \tag{16}$$

The coefficients of G are symmetric integer polynomials of the $a\beta_1, \cdots, a\beta_{2^n}$, and thus are integers. It follows that $g(z) = G(az)$ also has integer coefficients, and therefore so does f.

A similar but messier argument shows that h is also integer coefficient. Let

$$\begin{cases} H(w) = w^{p-1} G^p(w) , \\ J(w) = \dfrac{1}{w^p} \displaystyle\sum_{k=1}^{m} H(w + a\beta_k) = \dfrac{1}{w^p} \left[-rH(w) + \displaystyle\sum_{k=1}^{2^n} H(w + a\beta_k) \right] . \end{cases}$$

Noting that H has a zero of degree p at all of $a\beta_1, \cdots, a\beta_m$, it follows from the fundamental theorem that J is an integer coefficient polynomial. But then combining the definitions (11), (12), (15) and (16),

$$h(w) = \sum_{k=1}^{m} (w + \beta_k)^{p-1} g^p(w + \beta_k) = \frac{aw^p}{(aw)^p} \sum_{k=1}^{m} \left[(aw + a\beta_k)^{p-1} G^p(aw + a\beta_k) \right] = aw^p J(aw) .$$

It follows that h has integer coefficients.

References

[1] D. Hilbert, Ueber die Transcendenz der Zahlen e und π, *Math. Ann.*, **43** (1893) 216-219.

[2] M. Spivak, *Calculus* 4th ed., Publish or Perish, 2008.

[3] R. Steinberg and R. M. Redheffer, Analytic proof of the Lindemann theorem, *Pacific J. Math.*, **2** (1952) 231-242.

[4]The presentation in `http://www-users.math.umn.edu/~garrett/m/algebra/notes/15.pdf` is as nice as we've found, though the Wikipedia page on elementary symmetric polynomials is good enough. The theorem is standard and important, and not difficult to prove, but we are unaware of any particularly pretty proof.