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Abstract. We study the Fluctuation Theorem (FT) for entropy production in chaotic discrete-time dynamical systems on compact metric spaces, and extend it to empirical measures, all continuous potentials, and all weak Gibbs states. In particular, we establish the FT in the phase transition regime. These results hold under minimal chaoticity assumptions (expansiveness and specification) and require no ergodicity conditions. They are also valid for systems that are not necessarily invertible and involutions other than time reversal. Further extensions involve asymptotically additive potential sequences and the corresponding weak Gibbs measures. The generality of these results allows to view the FT as a structural facet of the thermodynamic formalism of dynamical systems.

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Contents

0 Introduction

This work concerns the mathematical theory of the so-called Fluctuation Relation (FR) and Fluctuation Theorem (FT) in the setting of discrete-time continuous dynamical systems on compact metric spaces. The FR is a universal property of the statistics of entropy production linked to time-reversal and the FT refers to a related Large Deviation Principle (LDP).

The discovery of the FR goes back to numerical experiments on the probability of violation of the 2$^{\mathrm{nd}}$ Law of Thermodynamics [ECM93] and associated theoretical works [ES94, GC95b, GC95a, Gal95] in the early 90's. In particular, the first formulation and mathematical proof of the FT were given in [GC95b] in the context of Anosov diffeomorphisms of compact Riemannian manifolds. Further steps in the mathematical development of the subject were taken in [Kur98, LS99, Mae99, Rue99]. These discoveries generated an enormous body of theoretical, numerical and experimental works which have fundamentally altered our understanding of non-equilibrium physics, with applications extending to chemistry and biology. For a review of these historical developments we refer the reader to [ES02, JQQ04, Gas05, RaM07, JPRB11] and to the forthcoming review articles [CJPS18, CJN$^+$18]; see also Example 0.7 below. The general mathematical structure and interpretation of the FR and FT from a modern point of view is briefly discussed in Section 1; see [CJPS18, CJN$^+$18] for additional information.

We shall consider dynamical systems (M, φ), where M is a compact metric space and $\varphi : M \to M$ is a continuous map. This is precisely the setting in which the FR and FT were initially discovered. We shall also assume that (M, φ) is chaotic in the sense that φ is expansive and satisfies Bowen's specification property.[1] We shall prove two sets of results. The first of them concerns the Periodic Orbits Fluctuation Principle (POFP). The second concerns the extension of the classical Gibbs type FR and FT to weak Gibbs measures, to which we shall refer as Gibbs Fluctuation Principle (GFP). Together they constitute a technical and conceptual extension of the previously known results on FR and FT. For example, the POFP holds for any continuous potential, and, more generally, for any asymptotically additive (not necessarily continuous) potential sequence. In the case of two-sided subshifts of finite type, the GFP holds for all Gibbs states (translation invariant or not) of any summable interaction Φ. The first result is new while the second (and only in part) was known to hold for interactions Φ satisfying Bowen's regularity assumption and, in particular, admitting a unique Gibbs state; see Example 0.5 below.

In the usual sense, the FT and FR are related to time reversal and require the map φ to be invertible. In this paper, we shall consider also involutions (other than time reversal) that do not require the invertibility of φ, and show that the FT and FR naturally extend to that case (see Section 3.2 for precise definitions).

We now describe some of our typical results. For simplicity, we consider in the remaining part of this introduction only the invertible case. We assume that φ is a homeomorphism, and introduce the following notion of *reversal map*: we assume that there is a continuous map $\theta : M \to M$ such that

$$\theta \circ \theta = \mathrm{Id}_M, \quad \varphi^{-1} = \theta \circ \varphi \circ \theta, \tag{0.1}$$

where Id_M stands for the identity mapping on M. Although in the main text of the paper our results are stated and proven in the general setting of the asymptotically additive thermodynamic formalism, we shall start with the familiar additive setting before turning to that level of generality.[2] We fix an arbitrary continuous function[3] $G : M \to \mathbb{R}$, and set

$$S_n G = G + G \circ \varphi + \cdots + G \circ \varphi^{n-1}.$$

We start with the POFP. Denote by M_n the set of n-periodic points of φ. Under our assumptions M_n is non-empty, finite, invariant under θ, and $\bigcup_n M_n$ is dense in M. We define a family of probability measures on M by

$$\mathbb{P}_n(\mathrm{d}y) = Z_n^{-1} \sum_{x \in M_n} \mathrm{e}^{S_n G(x)} \delta_x(\mathrm{d}y), \quad Z_n = Z_n(G) = \sum_{x \in M_n} \mathrm{e}^{S_n G(x)}, \tag{0.2}$$

where $n \geq 1$. Let

$$\widehat{\mathbb{P}}_n(\mathrm{d}y) = (\mathbb{P}_n \circ \theta)(\mathrm{d}y) = Z_n^{-1} \sum_{x \in M_n} \mathrm{e}^{S_n G \circ \theta(x)} \delta_x(\mathrm{d}y). \tag{0.3}$$

The measures $\widehat{\mathbb{P}}_n$ and $\mathbb{P}_n$ are absolutely continuous with respect to each other, and the logarithm of the corresponding density is given by

$$\log \frac{\mathrm{d}\mathbb{P}_n}{\mathrm{d}\widehat{\mathbb{P}}_n}(x) = S_n \sigma(x) \quad \text{for } x \in M_n,$$

[1] See [KH95, Section 18.3.c] and Remark 2.1 below. This assumption is made only for simplicity of exposition—all our results hold under a weaker assumption, see Section 2.1 for a precise statement.

[2] We shall freely use the standard notions of the usual thermodynamic formalism [Rue04, Wal82]. For the asymptotically additive extensions see [Bar11] and Section 2.

[3] Following the usual terminology, we shall often refer to G as a *potential*. The adjective *additive* refers to the property $S_{n+m}G = S_m G + S_n G \circ \varphi^m$ of the sequence $\{S_n G\}$.

where we write

$$\sigma = G - G \circ \theta$$

for the *entropy production observable*. Any weak limit point $\mathbb{P}$ of the sequence $\mathbb{P}_n$ is an equilibrium measure for G (see Proposition 2.8). Note that if $\mathbb{P}_{n_k} \rightharpoonup \mathbb{P}$, then $\widehat{\mathbb{P}}_{n_k} \rightharpoonup \widehat{\mathbb{P}} = \mathbb{P} \circ \theta$. The mathematical statement of the POFP is the Large Deviation Principle (LDP) for the empirical measures and ergodic averages of σ with respect to $\mathbb{P}_n$. Its interpretation, on which we shall elaborate in Section 1, quantifies the separation between $\widehat{\mathbb{P}}_{n_k}$ and $\mathbb{P}_{n_k}$, as these sequences of measures approach their limits $\widehat{\mathbb{P}}$ and $\mathbb{P}$.

Let $\mathcal{P}(M)$ be the set of all probability measures on M endowed with the topology of weak convergence. The following theorem summarizes the POFP.

Theorem A. *For any continuous function $G : M \to \mathbb{R}$, the following assertions hold.*

Large deviations. *There is a lower semicontinuous function $\mathbb{I} : \mathcal{P}(M) \to [0, +\infty]$ such that the sequence of empirical measures*

$$\mu_n^x = \frac{1}{n} \sum_{k=0}^{n-1} \delta_{\varphi^k(x)} \tag{0.4}$$

under the law $\mathbb{P}_n$ satisfies the LDP with the rate function $\mathbb{I}$.

Fluctuation theorem. *The sequence $\frac{1}{n} S_n \sigma$ under the law $\mathbb{P}_n$ satisfies the LDP with a rate function I given by the contraction of $\mathbb{I}$:*

$$I(s) = \inf \left\{ \mathbb{I}(\mathbb{Q}) : \mathbb{Q} \in \mathcal{P}(M), \int_M \sigma \mathrm{d}\mathbb{Q} = s \right\}. \tag{0.5}$$

Fluctuation relations. *The rate functions $\mathbb{I}$ and I satisfy the relations*

$$\mathbb{I}(\widehat{\mathbb{Q}}) = \mathbb{I}(\mathbb{Q}) + \int_M \sigma \mathrm{d}\mathbb{Q}, \qquad I(-s) = I(s) + s, \tag{0.6}$$

where $\mathbb{Q} \in \mathcal{P}(M)$, $s \in \mathbb{R}$ are arbitrary, and $\widehat{\mathbb{Q}} = \mathbb{Q} \circ \theta$.

Remark 0.1 The importance of periodic orbits for the study of chaotic dynamics in modern theory of dynamical systems goes back to seminal works of Bowen [Bow70] and Manning [Man71]. In the context of the FT and FR, periodic orbits played an important role in the early numerical works [ECM93]. Ruelle's proof of the Gallavotti–Cohen fluctuation theorem for Anosov diffeomorphisms [Rue99] was technically centered around periodic orbits. Further insights were obtained in [MV03] where, following the general scheme of [LS99, Mae99], the pairs $(\mathbb{P}_n, \widehat{\mathbb{P}}_n)$ and the entropy production observable σ were introduced, and the transient fluctuation relation was discussed. The work [MV03] primarily concerned Gibbs type FT for Bowen-regular potentials G, and we shall comment further on it in Example 0.7 below.

We now turn to the GFP. We shall assume that $\mathbb{P}$ is a weak Gibbs measure for some potential $G \in C(M)$[4] (see Definition 4.1 with $G_n = S_n G$).

Theorem B. *Let $\mathbb{P}$ be a weak Gibbs measure for a potential $G : M \to \mathbb{R}$. Then all three assertions formulated in Theorem A remain valid if we replace $\mathbb{P}_n$ with $\mathbb{P}$.*

[4] $C(M)/B(M)$ denotes the usual Banach space of continuous/bounded Borel real-valued functions on M.

Remark 0.2 On the technical level the key point of Theorems **A** and **B** is the LDP for the empirical measures (0.4), while the remaining properties are an easy consequence of it. The respective rate functions in Theorems **A** and **B** coincide.[5] The FR for the rate function $\mathbb{I}$ can be derived from an explicit formula, while the FR for I is implied by the contraction relation (0.5). For a more conceptual derivation of the FR for I see Section 1.1.

Remark 0.3 The LDP for empirical measures $\{\mu_n^x\}$ in Theorems **A** and **B** follows from a more general result that covers both cases and is stated and proven in Section 5. The proof of the LDP involves, as usual, two steps: the LD upper bound, which is a simple consequence of the existence of pressure (see Propositions 2.7 and 5.6), and the LD lower bound, which is more involved. A prototype of our argument appeared in the proofs of Theorem 3.1 in Föllmer–Orey [FO88] and Theorem 2.1 in Orey–Pelikan [OP88], in which the Shannon–McMillan–Breiman (SMB) theorem is used to derive the LD lower bound for Gibbs states of $\mathbb{Z}^d$ spin systems. By using Markov partitions, the same result was established for transitive Anosov diffeomorphisms on compact manifolds [OP89].[6] In our context, the SMB theorem is naturally replaced by its dynamical systems counterpart, the Brin–Katok local entropy formula [BK83]. The rest of our argument is related to the papers [You90, EKW94, PS05] (see also [PS18]). Although there the Brin–Katok theorem is not used directly, the key estimates entering Proposition 4.2 in [EKW94] and Proposition 3.1 in [PS05] are also important ingredients in the proof of the Brin–Katok formula and can be traced back to another work of Katok [Kat80, Theorem 1.1].

Remark 0.4 As Remarks 0.2 and 0.3 indicate, on the technical level Theorems **A** and **B** are closely related. We have separated them for historical reasons, for reasons of interpretation, and due to the role the specification plays in the proofs. Regarding the first two points, see Example 0.7 below and Section 1. Regarding the third one, in Theorem **B**, the specification is only needed to allow the use of Proposition 2.2, whereas in Theorem **A** a weak form of specification is crucial also in the proof of the lower bound of the LDP. There are alternative assumptions under which the conclusions of Proposition 2.2 can be established. For example, Pfister and Sullivan [PS05] prove it for dynamical systems with the so-called g-product property and apply it to β-shifts. Hence, Theorem **B** holds in that setting.

Before turning to the asymptotically additive setting, we briefly discuss several prototypical additive examples; see also Example 3.5 in Section 3.2. For the details and additional examples we refer the reader to the accompanying review article [CJPS18].

Example 0.5 (Two-sided subshift of finite type) Let[7] $\mathcal{A} = [\![1, \ell]\!]$ be a finite alphabet with discrete metric and let $\Omega = \mathcal{A}^{\mathbb{Z}}$ be the product space of two-sided sequences endowed with the usual metric

$$\Omega \times \Omega \ni (x, y) \mapsto d(x, y) = 2^{-\min\{j \in \mathbb{Z}_+ \,:\, x_j \neq y_j \text{ or } x_{-j} \neq y_{-j}\}}.$$

The shift operator $\varphi : \Omega \to \Omega$ defined by $\varphi(x)_j = x_{j+1}$ is obviously an expansive homeomorphism. We assume that (M, φ) is a subshift of finite type: given an $\ell \times \ell$ matrix $A = [A_{ij}]$ with entries $A_{ij} \in \{0, 1\}$ and such that all entries of the matrix A^m are strictly positive for some $m \geq 1$, one sets

$$M = \{x = (x_j)_{j \in \mathbb{Z}} : x_j \in \mathcal{A}, A_{x_j x_{j+1}} = 1 \text{ for all } j \in \mathbb{Z}\}.$$

[5] This fact is related to the *principle of regular entropic fluctuations* introduced in [JPRB11]; see Section 1.5.

[6] It is interesting to note that this result and the contraction principle immediately yield the Gallavotti–Cohen FT.

[7] Here and in the sequel $[\![1, \ell]\!] = [1, \ell] \cap \mathbb{Z}$.

In this case, φ is topologically mixing and satisfies Bowen's specification property. Let p be an involutive permutation of $\mathcal{A}$ and set $\theta(x)_j = p(x_{-j})$. Then θ is a homeomorphism of Ω satisfying (0.1). Thus, if in addition θ preserves M, then it is a reversal of (M, φ). For a subshift of finite type, that is the case whenever the adjacency matrix P associated to the map p commutes with A.

Theorem **A** applies to any G, and hence in situations where G exhibits phase transitions and the set of equilibrium states for G is not a singleton. Theorem **A** also covers the cases where G exhibits pathological behaviour from the phase transition point of view; see [Rue04, Section 3.17] and [Isr15, Section V.2]. For example, if $\mathcal{P} = \{\mathbb{P}_1, \cdots, \mathbb{P}_n\}$ is any finite collection of ergodic measure of the dynamical system (M, φ), then there exists a potential G whose set of ergodic equilibrium states is precisely $\mathcal{P}$. There is a dense set of G's in $C(M)$ with uncountably many ergodic equilibrium states. Although such general potentials could be considered non-physical, the POFP remains valid.

Regarding Theorem **B**, consider a spin chain whose set of allowed configurations is M, and let $\varPhi$ be a summable translation-invariant interaction. We shall follow the notation of the classical monograph [Rue04], and assume that $\varPhi$ belongs to the Banach space $\mathcal{B}$ of interactions introduced in Section 4.1 therein. We denote by $K_\varPhi \subset \mathcal{P}(M)$ the set of all Gibbs states for $\varPhi$. Then $K_\varPhi$ is a closed convex set and some elements of $K_\varPhi$ may not be φ-invariant. The set of φ-invariant elements of $K_\varPhi$ is precisely the set of equilibrium states for the potential $A_\varPhi$ (the contribution of one lattice site to the energy of a configuration) defined in Section 3.2 of [Rue04]. If $A_\varPhi$ satisfies Bowen's regularity condition (see [Bow74] and [KH95, Definition 20.2.5]), then $K_\varPhi$ is a singleton, but in general $K_\varPhi$ may have many distinct elements. However, it is not difficult to show that any $\mathbb{P} \in K_\varPhi$ is a weak Gibbs measure for the potential $A_\varPhi$ (see [EKW94, Lemma 3.2] and [CJPS18] for details), and Theorem **B** applies. These results extend to $\Omega = \mathcal{A}^{\mathbb{Z}^d}$ for any $d \geq 1$.

Example 0.6 (Uniformly hyperbolic systems) Let Ω be a compact connected Riemannian manifold and $\varphi : \Omega \to \Omega$ a C^1-diffeomorphism. Let $M \subset \Omega$ be a locally maximal invariant hyperbolic set such that $\varphi|_M$ is transitive. Then the map φ is an expansive homeomorphism of M satisfying Bowen's specification property. Hence Theorems **A** and **B** hold for (M, φ); see [Bow75, PP90].

Example 0.7 (Anosov diffeomorphisms) Continuing with the previous example, if $M = \Omega$, then (Ω, φ) is a transitive Anosov system. This is the original setting in which the first FR and FT were proven.

We denote by $D(x) = |\det \varphi'(x)|$ the Jacobian of φ at x and set

$$D^{s/u}(x) = |\det(\varphi'(x)|_{E_x^{s/u}})|,$$

where $E_x^{s/u}$ denotes the stable/unstable tangent subspace at $x \in M$. The C^1-regularity of φ implies that the maps

$$x \mapsto D(x), \qquad x \mapsto D^{s/u}(x), \tag{0.7}$$

are continuous. The potential

$$G(x) = -\log D^u(x) \tag{0.8}$$

is of particular importance [ER85, EP86], and in the context of FR its relevance goes back to the pioneering work [ECM93]. As a special case of Example 0.6, Theorem **A** holds for this G and any continuous reversal θ. Theorem **B** holds for any weak Gibbs measure for G.

If φ is $C^{1+\alpha}$ for some $\alpha > 0$, then the maps (0.7) are Hölder continuous and the potential G has a unique equilibrium state, the SRB probability measure $\mathbb{P}_{\rm srb}$. In this case, denoting by $\mathbb{P}_{\rm vol}$ the normalized Riemannian volume measure on M, the empirical measures (0.4) converge weakly to $\mathbb{P}_{\rm srb}$ for $\mathbb{P}_{\rm vol}$-a.e. $x \in M$. The measure $\mathbb{P}_{\rm srb}$ enjoys very strong ergodic properties and, in particular, is weak Gibbs for G, and Theorem **B** applies to $\mathbb{P}_{\rm srb}$. In this case, the LDP part of Theorem **B** goes back to [OP89]. Since $\mathbb{P}_{\rm vol}$ is also a weak Gibbs measure for G,[8] Theorem **B** applies to $\mathbb{P}_{\rm vol}$ as well.

Example 0.8 (Anosov diffeomorphisms: historical perspective) The original formulation of the Gallavotti–Cohen FT [GC95b, GC95a] concerns $C^{1+\alpha}$ transitive Anosov diffeomorphisms with the additional assumption that the reversal map θ is C^1. The entropy production observable is taken to be the phase space contraction rate

$$\widetilde{\sigma}(x) = -\log D(x), \tag{0.9}$$

and the LDP concerns the time averages $n^{-1}S_n(\widetilde{\sigma})$. Since θ is C^1, the tangent map $\theta'(x)$ provides an isomorphism between $E_x^{s/u}$ and $E_{\theta(x)}^{u/s}$, and

$$\log D^u \circ \theta = -\log D^s \circ \varphi^{-1}. \tag{0.10}$$

As observed in [MV03], this relation gives that for some $C > 0$, all $x \in M$ and all n,

$$|S_n\widetilde{\sigma}(x) - S_n\sigma(x)| < C,$$

where $\sigma = G - G \circ \theta$ with G given by (0.8). Hence, under the assumptions of [GC95b, GC95a], the Gallavotti–Cohen FT and the FT of Theorem **B** are identical statements.

The assumption that θ is C^1 is essential for the Gallavotti–Cohen FT. Porta [Por10] has exhibited examples of C^ω Anosov diffeomorphisms on the torus $\mathbb{T}^2$ which admit continuous but not differentiable reversals, and for which the Gallavotti–Cohen FT *fails* in the sense that the LDP rate function for the averages $n^{-1}S_n(\widetilde{\sigma})$ does not satisfy the second relation in (0.6). For his examples Porta also identifies the entropy production observable $\sigma = G - G \circ \theta$ noticing that the LDP holds for it with a rate function satisfying the FR (0.6).

Porta's observation was a rediscovery of an important insight of Maes and Verbitskiy. Returning to our general setting (M, φ), in [MV03] the entropy production observable $\sigma = G - G \circ \theta$ is introduced for an arbitrary potential G, and the Gibbs FR and FT were established for the averages $n^{-1}S_n(\sigma)$ assuming that G satisfies the Bowen regularity condition. In this case $\mathbb{P}$ is again the unique equilibrium measure for G and enjoys very strong ergodic properties. The proofs of [MV03] are further simplified in [JPRB11]; see Section 1.3 below.

We now turn to the asymptotically additive setting.

Definition 0.9 *A sequence of functions $\mathcal{G} = \{G_n\}_{n\geq 1} \subset B(M)$ is called* asymptotically additive *if there is a sequence $\{G^{(k)}\}_{k\geq 1} \subset C(M)$ such that*

$$\lim_{k\to\infty} \limsup_{n\to\infty} n^{-1}\big\|G_n - S_nG^{(k)}\big\|_\infty = 0. \tag{0.11}$$

The set of all asymptotically additive sequences of functions on M is denoted by $\mathcal{A}(M)$, and a family $\{G^{(k)}\}_{k\geq 1} \subset C(M)$ satisfying (0.11) is called an approximating sequence[9] *for $\mathcal{G}$.*

[8] This follows from the Volume Lemma; see [Bow75, Lemma 4.7] and [KH95, Lemma 20.4.2].

[9] Note that G_n is not required to be continuous, but $G^{(k)}$ is. The notion of asymptotically additive potential was first introduced in [FH10], and there G_n is required to be continuous (see Section 6 for a detailed discussion of this point).

Except in Section 5.3, the elements of $\mathcal{A}(M)$ will play the role of *potentials*, and hence we shall often refer to them as *asymptotically additive potential sequences*.

Remark 0.10 An obvious example of an asymptotically additive potential sequence is $\mathcal{G} = \{S_n G\}$, where $G \in C(M)$. We shall refer to this special case as *additive*. Some other conditions, which either imply asymptotic additivity or are equivalent to it, are given in Theorem 6.1. There, we prove in particular that if $\mathcal{G} = \{S_n G\}$ (with $G \in B(M)$) satisfies the *tempered variation* condition (which is weaker than the continuity of G), then $\mathcal{G}$ is asymptotically additive. The tempered variation condition, which to the best of our knowledge goes back to [Kes01] (see also [Bar06]), holds in particular if G satisfies the *bounded variation* condition of [Rue92]. Another class of examples is given by *weakly almost additive* potentials,[10] which are characterized by the following property: there is a sequence $\{C_n\}_{n\geq 1} \subset \mathbb{R}$ such that $\lim_{n\to\infty} n^{-1}C_n = 0$ and

$$-C_m + G_m + G_n \circ \varphi^m \leq G_{m+n} \leq C_m + G_m + G_n \circ \varphi^m, \quad m, n \geq 0. \tag{0.12}$$

If a family $\mathcal{G} \subset C(M)$ is weakly almost additive, then it is asymptotically additive with $G^{(k)} = k^{-1}G_k$; see Lemma 6.2.

Remark 0.11 Note that $\mathcal{A}(M)$ is a vector space and that

$$\|\mathcal{G}\|_* := \limsup_{n\to\infty} n^{-1}\|G_n\|_\infty \tag{0.13}$$

defines a seminorm on $\mathcal{A}(M)$, which in turn gives a natural equivalence relation: $\mathcal{G} \sim \mathcal{G}'$ iff $\|\mathcal{G} - \mathcal{G}'\|_* = 0$ (finiteness of (0.13) follows immediately from (0.11)). As mentioned in [FH10, Remark A.6 (ii)] (see also the beginning of Section 3.2 in [BV15]), equivalent potential sequences share many important properties and, in particular, have the same approximating sequences. Furthermore, if $V, V' \in C(M)$ are such that $V' = V + U - U \circ \varphi$ for some $U \in C(M)$, then $\{S_n V\} \sim \{S_n V'\}$, so that this concept of equivalence generalizes the standard notion of equivalence for potentials. Moreover, by the definition of asymptotic additivity, for all $\mathcal{G} \in \mathcal{A}(M)$ we have $\lim_{k\to\infty} \|\mathcal{G} - \{S_n G^{(k)}\}\|_* = 0$, so that the additive potential sequences are dense in the quotient space $\mathcal{A}(M)/\!\sim$. Finally, we note that in each equivalence class there exists an asymptotically additive potential sequence that consists of *continuous* functions (see Remark 6.5).

Remark 0.12 To the best of our knowledge, the first extension of the classical thermodynamic formalism of Ruelle and Walters [Rue04, Wal82] beyond the additive setting goes back to the work of Falconer [Fal88]. This and later extensions were principally motivated by the multifractal analysis of certain classes of self-similar sets, and in this context the subject has developed rapidly; see for example [CFH08, FH10, ZZC11, Bar11, VZ15, IY17] and references therein.

It is likely that the subject will continue to flourish with an expanding number of applications that cannot be reached with classical theory; see Example 0.14 below and recent works [BJPP17, BCJP18] for applications to theory of repeated quantum measurement processes.

Theorems **A** and **B** extend to asymptotically additive potential sequences with the following notational changes. Given $\mathcal{G} = \{G_n\} \in \mathcal{A}(M)$, one defines a sequence of probability measures on M by the relations (cf. (0.2))

$$\mathbb{P}_n(\mathrm{d}y) = Z_n^{-1} \sum_{x\in M_n} \mathrm{e}^{G_n(x)}\delta_x(\mathrm{d}y), \quad Z_n = Z_n(\mathcal{G}) = \sum_{x\in M_n} \mathrm{e}^{G_n(x)}. \tag{0.14}$$

[10] In the literature, the special case where $C_n = C$ is often called *almost additive*, and we shall use this convention in the sequel; see [Bar11].

The time-reversal operation is now defined as $\theta_n = \theta \circ \varphi^{n-1}$. Let us set

$$\widehat{\mathbb{P}}_n(\mathrm{d}y) = (\mathbb{P}_n \circ \theta_n)(\mathrm{d}y) = Z_n^{-1} \sum_{x \in M_n} \mathrm{e}^{G_n \circ \theta \circ \varphi^{n-1}(x)} \delta_x(\mathrm{d}y) \tag{0.15}$$

and remark that this relation coincides with (0.3) in the case of additive potentials. We also note that

$$\log \frac{\mathrm{d}\mathbb{P}_n}{\mathrm{d}\widehat{\mathbb{P}}_n}(x) = \sigma_n(x) \quad \text{for } x \in M_n,$$

where we write

$$\sigma_n = G_n - G_n \circ \theta_n \tag{0.16}$$

for the *entropy production in time* n. Accordingly, the ergodic averages $n^{-1}S_n\sigma$ are now replaced by $n^{-1}\sigma_n$. With the above notational changes Theorems **A** and **B** hold for any $\mathcal{G} \in \mathcal{A}(M)$. Starting with Section 2 we shall work exclusively in the asymptotically additive setting.

Example 0.13 (Boundary terms in spin chains) Consider the spin chains discussed in Example 0.5 in the case where M is the full two-sided shift (*i.e.*, when $A_{ij} = 1$ for all i, j). The measures $\mathbb{P}_n$ of Theorem **A** can be interpreted as Gibbs measures on finite-size systems, since to each configuration of a system of n spins, one can associate an orbit of period n in M and vice versa. The case $G_n = S_n A_\Phi$ then corresponds to periodic boundary conditions, while arbitrary boundary conditions lead to asymptotically additive sequences of the kind $G_n = S_n A_\Phi + g_n$, where the boundary term g_n satisfies $\lim_{n\to\infty} n^{-1}\|g_n\|_\infty = 0$.

Example 0.14 (Matrix product potentials) Perhaps the best known examples of asymptotically additive potential sequences arise through matrix products. Let $\mathcal{M} : M \to \mathbb{M}_N(\mathbb{C})$ be a continuous map such that $\|\mathcal{M}(x)\mathcal{M}(\varphi(x))\cdots\mathcal{M}(\varphi^{n-1}(x))\| \neq 0$ for all $n \geq 1$ and $x \in M$ (here $\mathbb{M}_N$ is the vector space of complex $N \times N$ matrices). The potential sequences of the form

$$G_n(x) = \log \|\mathcal{M}(x)\mathcal{M}(\varphi(x))\cdots\mathcal{M}(\varphi^{n-1}(x))\| \tag{0.17}$$

arise in multifractal analysis of self-similar sets, see for example [FO03, Fen03, OST05, Fen09, Bar11]. Sequences of this type also describe the statistics of some important classes of repeated quantum measurement processes [BJPP17, BCJP18]. Except in trivial cases, the sequence $\{G_n\}$ is not additive. Note that the upper almost additivity

$$G_{n+m} \leq C + G_m + G_n \circ \varphi^m \tag{0.18}$$

always holds with a constant C that depends only on the choice of the matrix norm on $\mathbb{M}_N(\mathbb{C})$ appearing in (0.17). If the matrix entries of $\mathcal{M}(x)$ are strictly positive for all $x \in M$, or if $N = 2$ and $\mathcal{M}$ satisfies the cone condition of [BG06][11] (in the context of nonconformal repellers), then one can show that

$$-C + G_m + G_n \circ \varphi^m \leq G_{n+m} \tag{0.19}$$

for some $C > 0$, so that the sequence $\{G_n\}$ is almost additive. In many interesting examples, however, (0.19) fails, but $\mathcal{G}$ remains asymptotically additive and hence our results apply. When the potential defined in (0.17) is not asymptotically additive, it can exhibit a very singular behaviour from the thermodynamic formalism point of view.[12] For reasons of space we postpone the discussion of the last point to the forthcoming articles [CJPS17, BCJP18].

[11] This type of condition can be traced back to [Rue79]. See also [Bar11, Definition 11.2.1]

[12] In some cases, this singularity is dictated by the number theoretic properties of the entries of $\mathcal{M}(x)$; see [BCJP18] for a discussion.

Example 0.15 Let (M, φ) and θ be as in Example 0.5, and let $\mathbb{P}$ be any fully-supported invariant measure on M. For all $n \geq 1$, define $G_n \in C(M)$ by $G_n(x) = \log \mathbb{P}\{y \in M : y_i = x_i, i = 1, \dots, n\}$, where $x = (x_1, x_2, \dots)$. Then, if $\mathcal{G} = \{G_n\}$ is asymptotically additive, the measure $\mathbb{P}$ is weak Gibbs with respect to $\mathcal{G}$, and hence Theorem **B** applies. In this setup, (0.12) is interpreted as "weak dependence." In particular, all invariant quasi-Bernoulli measures (*i.e.*, satisfying (0.18) and (0.19)) on M are weak Gibbs with respect to an asymptotically additive potential.

We finish with the following general remarks.

Remark 0.16 To summarize, the contribution of our paper is two-fold. Firstly, to the best of our knowledge, the POFP has not appeared previously in the literature and provides rather general formulation and proof of the FT and FR in the context of chaotic dynamical systems on compact metric spaces. Furthermore, the GFP extends the FT and FR of [MV03] to weak Gibbs measures (which do not even need to be invariant). In particular, this extends the validity of the FT and FR to the phase transition regime, as illustrated in Example 0.5. Both results hold for any asymptotically additive potential sequence. Secondly, the FR for the rate function of empirical measures (the first relation in (0.6)) is new and we plan to investigate it further in other models of relevance to non-equilibrium statistical mechanics. Let us mention, however, that in the context of Markov jump processes, the FR for trajectories and the related LDP contraction were previously discussed in [BC15, Section 5].

Remark 0.17 To the best of our knowledge, the FT and FR in the phase transition regime have not been previously discussed in the physics and mathematics literature, apart from stochastic lattice gases; see [BDG+06, BDG+15]. On the other hand, a considerable amount of efforts in the dynamical systems community over the last two decades has been devoted to the extension of multifractal analysis to the phase transition regime; for instance, see [Mak98, FO03, Tes06]. Given the link between multifractal analysis and large deviations theory [DK01, Kes01], the two research directions are related, and this connection remains to be investigated in the future.

Remark 0.18 Although the conceptual emphasis of this paper has been on the FT and FR for entropy production generated by a reversal operation, the LDP parts of Theorems **A** and **B** are of independent interest and have wider applicability; see [CJPS17] for a more general approach than the one adopted here, and [BCJP18] for some concrete applications in the context of repeated quantum measurement processes.

The paper is organized as follows. Section 1 is a continuation of the introduction where we review the general mathematical structure and interpretation of the FR and FT from a modern point of view. In Section 2 we collect preliminaries needed for the formulations and proofs of our results, including an overview of the asymptotically additive thermodynamic formalism. Section 3 is devoted to the POFP and Section 4 to the GFP. Our main technical results regarding the LDP are stated and proven in Section 5. Finally, in Section 6 we discuss some properties and characterizations of asymptotically additive potential sequences.

This work is accompanied by a review article [CJPS18] where the reader can find additional information and examples regarding the FT and FR.

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1 Prologue: what is the Fluctuation Theorem?

1.1 Transient fluctuation relations

Our starting point is a family of probability spaces $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$ indexed by a parameter $n \in \mathbb{N}$. Each of these spaces is equipped with a measurable involution $\Theta_n : \Omega_n \to \Omega_n$ called *reversal* (the mapping Θ_n is its own inverse). In many cases of interest the probability space $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$ describes the space-time statistics of the physical system under consideration over the finite time interval $[0, n]$, and the map Θ_n is related to time-reversal.

Let us set $\widehat{\mathbb{P}}_n = \mathbb{P}_n \circ \Theta_n$ and impose the following hypothesis:

(R) Regularity. *The measures $\widehat{\mathbb{P}}_n$ and $\mathbb{P}_n$ are equivalent.*

Under Assumption **(R)**, one defines

$$\sigma_n = \log \frac{\mathrm{d}\mathbb{P}_n}{\mathrm{d}\widehat{\mathbb{P}}_n}. \tag{1.1}$$

This is a real-valued random variable on Ω_n, and we denote by P_n its law under $\mathbb{P}_n$. The very definition of σ_n implies a number of simple, yet important properties.

Relative entropy. The relative entropy[13] of $\mathbb{P}_n$ with respect to $\widehat{\mathbb{P}}_n$ is given by the relation

$$\mathrm{Ent}(\mathbb{P}_n \,|\, \widehat{\mathbb{P}}_n) \equiv \int_{\Omega_n} \log\left(\frac{\mathrm{d}\mathbb{P}_n}{\mathrm{d}\widehat{\mathbb{P}}_n}\right) \mathrm{d}\mathbb{P}_n = \int_{\Omega_n} \sigma_n \mathrm{d}\mathbb{P}_n = \int_{\mathbb{R}} s\, P_n(\mathrm{d}s).$$

Since this quantity is non-negative, we obtain

$$\int_{\mathbb{R}} s\, P_n(\mathrm{d}s) \geq 0, \tag{1.2}$$

which in the physics literature is sometimes called *Jarzynski's inequality*. This inequality asserts that under the law $\mathbb{P}_n$, positive values of σ_n are favored.

Rényi entropy. Rényi's relative α-entropy of $\widehat{\mathbb{P}}_n$ with respect to $\mathbb{P}_n$ is defined by

$$\mathrm{Ent}_\alpha(\mathbb{P}_n \,|\, \widehat{\mathbb{P}}_n) \equiv \log \int_{\Omega_n} \left(\frac{\mathrm{d}\widehat{\mathbb{P}}_n}{\mathrm{d}\mathbb{P}_n}\right)^\alpha \mathrm{d}\mathbb{P}_n = \log \int_{\Omega_n} \mathrm{e}^{-\alpha\sigma_n} \mathrm{d}\mathbb{P}_n = \log \int_{\mathbb{R}} \mathrm{e}^{-\alpha s} P_n(\mathrm{d}s) \equiv e_n(\alpha).$$

[13] Also called Kullback–Leibler divergence.

The function $\mathbb{R} \ni \alpha \mapsto e_n(\alpha) \in]-\infty, +\infty]$ is convex and lower semicontinuous. It vanishes at $\alpha = 0$ and $\alpha = 1$, so that $e_n(\alpha)$ is non-positive and finite on $[0, 1]$, and non-negative outside $[0, 1]$. It admits an analytic continuation to the strip $\{z \in \mathbb{C} : 0 < \mathrm{Re}\, z < 1\}$ which is continuous on its closure. Expressing the relation $e_n(1) = 0$ in terms of P_n, we derive

$$\int_{\mathbb{R}} \mathrm{e}^{-s} P_n(\mathrm{d}s) = 1. \tag{1.3}$$

In the physics literature this relation is sometimes called *Jarzynski's identity*.

Proposition 1.1 *In the above setting, the following two relations hold:*

$$e_n(\alpha) = e_n(1-\alpha) \quad \text{for } \alpha \in \mathbb{R}, \tag{1.4}$$

$$\frac{\mathrm{d}P_n}{\mathrm{d}\widehat{P}_n}(s) = \mathrm{e}^s \quad \text{for } s \in \mathbb{R}, \tag{1.5}$$

where $\widehat{P}_n$ is the image of P_n under the reflection $\vartheta(s) = -s$.

Remark 1.2 Relations (1.4) and (1.5) are in fact equivalent: the validity of one of them implies the other. We shall refer to them as the *transient FR*. It refines (and implies) the conclusion of the Jarzynski inequality (1.2) and its basic appeal is its universal form. In applications to non-equilibrium physics, the transient FR is a fingerprint of time-reversal symmetry breaking and emergence of the 2$^{\text{nd}}$ Law of Thermodynamics.

Proof of Proposition 1.1. Relation (1.4) is a simple consequence of a symmetry property of Rényi's entropy:

$$\begin{aligned} e_n(1-\alpha) &= \mathrm{Ent}_{1-\alpha}(\mathbb{P}_n \,|\, \widehat{\mathbb{P}}_n) = \mathrm{Ent}_\alpha(\widehat{\mathbb{P}}_n \,|\, \mathbb{P}_n) = \log \int_{\Omega_n} \mathrm{e}^{\alpha\sigma_n} \mathrm{d}\widehat{\mathbb{P}}_n \\ &= \log \int_{\Omega_n} \mathrm{e}^{\alpha\sigma_n \circ \Theta_n} \mathrm{d}(\widehat{\mathbb{P}}_n \circ \Theta_n) = \log \int_{\Omega_n} \mathrm{e}^{-\alpha\sigma_n} \mathrm{d}\mathbb{P}_n = e_n(\alpha), \end{aligned}$$

where we used the elementary relation $\sigma_n \circ \Theta_n = -\sigma_n$.

To prove (1.5), we exponentiate (1.4) and rewrite the result in terms of P_n:

$$\int_{\mathbb{R}} \mathrm{e}^{-\alpha s} P_n(\mathrm{d}s) = \int_{\mathbb{R}} \mathrm{e}^{-(1-\alpha)s} P_n(\mathrm{d}s) = \int_{\mathbb{R}} \mathrm{e}^{-\alpha s} (\mathrm{e}^s \widehat{P}_n)(\mathrm{d}s).$$

Using now the analyticity of the function $e_n(z)$ in the open strip $0 < \mathrm{Re}\, z < 1$, its continuity in the closed strip $0 \le \mathrm{Re}\, z \le 1$, and the fact that the characteristic function uniquely defines the corresponding measure, we deduce that $P_n(\mathrm{d}s)$ and $(\mathrm{e}^s \widehat{P}_n)(\mathrm{d}s)$ coincide. This is equivalent to (1.5). □

1.2 Fluctuation Theorem and Fluctuation Relation

Definition 1.3 *We shall say that the* Fluctuation Theorem *holds for the family* $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n, \Theta_n)$ *if there is a lower semicontinuous function* $I : \mathbb{R} \to [0, +\infty]$ *such that, for any Borel set* $\Gamma \subset \mathbb{R}$,

$$\begin{aligned} -\inf_{s \in \dot{\Gamma}} I(s) &\le \liminf_{n\to\infty} n^{-1} \log \mathbb{P}_n\{n^{-1}\sigma_n \in \Gamma\} \\ &\le \limsup_{n\to\infty} n^{-1} \log \mathbb{P}_n\{n^{-1}\sigma_n \in \Gamma\} \le -\inf_{s\in\overline{\Gamma}} I(s), \end{aligned} \tag{1.6}$$

where $\dot{\Gamma}/\overline{\Gamma}$ *denotes the interior/closure of* Γ.

Let us note that $\mathbb{P}_n\{n^{-1}\sigma_n \in \Gamma\} = P_n(n\Gamma)$, so that (1.6) can be rewritten as

$$-\inf_{s\in\dot{\Gamma}} I(s) \le \liminf_{n\to\infty} n^{-1}\log P_n(n\Gamma) \le \limsup_{n\to\infty} n^{-1}\log P_n(n\Gamma) \le -\inf_{s\in\overline{\Gamma}} I(s). \tag{1.7}$$

The following result shows that the transient FR implies a symmetry relation for the rate function in the FT.

Proposition 1.4 *Suppose that the FT holds for a family* $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n, \Theta_n)$. *Then the corresponding rate function* I *satisfies the relation*

$$I(-s) = I(s) + s \quad \textit{for } s \in \mathbb{R}. \tag{1.8}$$

Proof. In view of (1.5), for any Borel set $\Gamma \subset \mathbb{R}$ we have

$$P_n(\Gamma) \le \mathrm{e}^{\sup\Gamma} P_n(-\Gamma).$$

Replacing Γ with $n\Gamma$ and using (1.7) we see that

$$-\inf_{s\in\dot{\Gamma}} I(s) \le \liminf_{n\to\infty} n^{-1}\log P_n(n\Gamma) \le \limsup_{n\to\infty} n^{-1}\log\bigl(e^{n\sup\Gamma}P_n(-n\Gamma)\bigr) \le \sup\Gamma - \inf_{s\in\overline{\Gamma}} I(-s).$$

Taking $\Gamma =]a-\epsilon, a+\epsilon[$ with $\epsilon > 0$, we derive

$$\inf_{|s+a|<2\epsilon} I(s) \le \inf_{|s+a|\le\epsilon} I(s) \le a + \epsilon + \inf_{|s-a|<\epsilon} I(s). \tag{1.9}$$

Since the function I is lower semicontinuous, we have $I(a) = \lim_{\epsilon\downarrow 0}\inf_{|s-a|<\epsilon} I(s)$. Passing to the limit in (1.9) as $\epsilon \downarrow 0$, we obtain

$$I(-a) \le a + I(a)$$

for any $a \in \mathbb{R}$. Replacing a by $-a$ and comparing the two inequalities, we arrive at (1.8). □

1.3 Entropic pressure

Suppose that the Fluctuation Theorem (Definition 1.3) holds. Then under very general conditions Varadhan's lemma implies that the limit

$$e(\alpha) = \lim_{n\to\infty} n^{-1} e_n(\alpha) \tag{1.10}$$

exists for all $\alpha \in \mathbb{R}$ and that

$$e(\alpha) = -\inf_{s\in\mathbb{R}}\bigl(s\alpha + I(s)\bigr) = \sup_{s\in\mathbb{R}}\bigl(-s\alpha - I(s)\bigr). \tag{1.11}$$

The function $e(\alpha)$ is called the *entropic pressure* of the family $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n, \Theta_n)$. Elementary arguments show that:

(a) the function $e(\alpha)$ is convex and lower semicontinuous;

(b) it is non-positive on $[0,1]$ and non-negative outside $[0,1]$, with a global minimum at $\alpha = 1/2$;

(c) it satisfies the relations $e(0) = e(1) = 0$ and

$$e(\alpha) = e(1-\alpha) \quad \text{for all } \alpha \in \mathbb{R}. \tag{1.12}$$

If the rate function $I(s)$ is convex (which is the case in the setting of this paper), then inverting the Legendre transform (1.11) gives[14]

$$I(s) = -\inf_{\alpha\in\mathbb{R}}\big(s\alpha + e(\alpha)\big) = \sup_{\alpha\in\mathbb{R}}\big(s\alpha - e(-\alpha)\big). \tag{1.13}$$

The above discussion can be turned around. Suppose that limit (1.10) exists and that $e(\alpha)$ is differentiable on $\mathbb{R}$. Then the Gärtner–Ellis theorem (see Section 2.3 in [DZ00]) implies that the Fluctuation Theorem holds with the convex lower semicontinuous rate function I given by (1.13). This gives a technical route to prove the Fluctuation Theorem. Since the seminal work [LS99] this route has been dominant in mathematical approaches to the FT.

Returning to the dynamical system (M, φ), the above route yields a quick proof of Theorem **A** if the potentials G and $G \circ \theta$ are Bowen-regular. We follow [JPRB11]. By a classical result of Bowen, the limit

$$e(\alpha) = \lim_{n\to\infty} \frac{1}{n} \log \int_{M_n} e^{-\alpha S_n(\sigma)} \mathrm{d}\mathbb{P}_n$$

exists for all $\alpha \in \mathbb{R}$ and is equal to the topological pressure of the potential $(1-\alpha)G + \alpha G \circ \theta$; see also Theorem 2.7 in Section 2.4. By another classical result of Bowen [Bow75], the potential $(1-\alpha)G + \alpha G \circ \theta$ has unique equilibrium state for all α, and [Wal82, Theorem 9.15] shows that $e(\alpha)$ is differentiable on $\mathbb{R}$. Thus, the Gärtner–Ellis theorem applies and gives the FT and the second FR in (0.6). Assuming that the vector space of all Bowen-regular potentials in dense in $C(M)$ (for instance, this is the case in Examples 0.5–0.7 of the introduction), Kifer's theorem [Kif90] implies that the LDP part of Theorem **A** holds. In this case the first FR in (0.6) follows from a computation given in Section 3.2. The same proof applies verbatim to Theorem **B**, and in particular recovers the results of [MV03].

The novelty of Theorems **A** and **B** is that they hold for potentials G for which the entropic pressure is not necessarily differentiable, hence in the phase transition regime. In this case the proof follows a different strategy: one first proves the LDP for empirical measures, and then uses the contraction principle to prove the FT. Another novelty is that these results also extend to asymptotically additive potential sequences.

1.4 What does the Fluctuation Theorem mean?

Returning to the level of generality of Sections 1.1 and 1.2, the interpretation of the rate function I in the FT is given in terms of *hypothesis testing error exponents* of the family $\{(\mathbb{P}_n, \widehat{\mathbb{P}}_n)\}$. These exponents describe the rate of separation between measures $\mathbb{P}_n$ and $\widehat{\mathbb{P}}_n$ as $n \to \infty$. If the elements of $\mathbb{N}$ are instances of time and Θ_n is related to time-reversal, these exponents quantify the emergence of the arrow of time and can be viewed as a fine form of the second law of thermodynamics. Let us recall the definition of three exponents relevant to our study and state some results without proofs, which can be found in [JOPS12, CJN+18].

[14] If I is not convex on $\mathbb{R}$, the same relations holds if I is replaced by its convex envelope.

Stein error exponent. *Given* $\gamma \in]0,1[$, *we set*

$$\mathfrak{s}_\gamma(\mathbb{P}_n, \widehat{\mathbb{P}}_n) = \inf\big\{\widehat{\mathbb{P}}_n(\Gamma) : \Gamma \in \mathcal{F}_n, \widehat{\mathbb{P}}_n(\Gamma^c) \leq \gamma\big\}. \tag{1.14}$$

The following result establishes a link between the large n asymptotics of $\mathfrak{s}_\gamma(\mathbb{P}_n, \widehat{\mathbb{P}}_n)$ and the weak law of large numbers.

Proposition 1.5 *Suppose that* $n^{-1}\sigma_n$ *converges in probability*[15] *to a deterministic limit* ep. *Then*

$$\lim_{n\to\infty} n^{-1} \log \mathfrak{s}_\gamma(\mathbb{P}_n, \widehat{\mathbb{P}}_n) = -\mathrm{ep} \quad \textit{for any } \gamma \in]0,1[. \tag{1.15}$$

Moreover, the above limit coincides with the following (equal) quantities:

$$\underline{\mathfrak{s}} = \inf\Big\{\liminf_{n\to\infty} n^{-1} \log \widehat{\mathbb{P}}_n(\Gamma_n) : \Gamma_n \in \mathcal{F}_n, \lim_{n\to\infty} \mathbb{P}_n(\Gamma_n^c) = 0\Big\},$$
$$\overline{\mathfrak{s}} = \inf\Big\{\limsup_{n\to\infty} n^{-1} \log \widehat{\mathbb{P}}_n(\Gamma_n) : \Gamma_n \in \mathcal{F}_n, \lim_{n\to\infty} \mathbb{P}_n(\Gamma_n^c) = 0\Big\}.$$

If in addition the entropic pressure $e(\alpha)$ *is differentiable at* $\alpha = 0$, *then*

$$\lim_{n\to\infty} n^{-1}\mathrm{Ent}(\mathbb{P}_n|\widehat{\mathbb{P}}_n) = \mathrm{ep} = -e'(0).$$

Recall that, for two probability measures $\mathbb{P}$ and $\mathbb{Q}$ on a measurable space $(\Omega, \mathcal{F})$, the *total variation distance* is given by

$$\|\mathbb{P} - \mathbb{Q}\|_{\mathrm{var}} = \sup_{\Gamma\in\mathcal{F}} |\mathbb{P}(\Gamma) - \mathbb{Q}(\Gamma)| = 1 - \int_\Omega (\Delta \wedge 1)\mathrm{d}\mathbb{Q},$$

where the second relation holds if $\mathbb{P}$ is absolutely continuous with respect to $\mathbb{Q}$, and Δ stands for the corresponding density.

Chernoff error exponents. *The lower and upper Chernoff exponents are defined by*

$$\underline{c} = \liminf_{n\to\infty} n^{-1} \log\big(1 - \|\mathbb{P}_n - \widehat{\mathbb{P}}_n\|_{\mathrm{var}}\big), \quad \overline{c} = \limsup_{n\to\infty} n^{-1} \log\big(1 - \|\mathbb{P}_n - \widehat{\mathbb{P}}_n\|_{\mathrm{var}}\big). \tag{1.16}$$

The following result provides a sufficient condition for these two quantities to be equal and expresses them in terms of the entropic pressure.

Proposition 1.6 *Suppose that the FT holds with the rate function* (1.13). *Then the upper and lower Chernoff exponents coincide, and*

$$\underline{c} = \overline{c} = e(1/2). \tag{1.17}$$

Let us note that $e(1/2) = -I(0)$. Thus, if $I(0) > 0$, then the measures $\mathbb{P}_n$ and $\widehat{\mathbb{P}}_n$ concentrate on the complementary subsets $\{\sigma_n > 0\}$ and $\{\sigma_n < 0\}$, respectively, and separate with an exponential rate $-I(0)$:

$$\lim_{n\to\infty} n^{-1} \log \mathbb{P}_n\{\sigma_n < 0\} = \lim_{n\to\infty} n^{-1} \log \widehat{\mathbb{P}}_n\{\sigma_n > 0\} = -I(0).$$

[15] For instance, if the FT holds and the corresponding rate function I vanishes at a unique point $s_0 \in \mathbb{R}$, then (1.15) holds with $\mathrm{ep} = s_0$.

Hoeffding error exponent. *Given $\vartheta \in \mathbb{R}$, we define*

$$\underline{\mathfrak{h}}_\vartheta = \inf\Big\{\liminf_{n\to\infty} n^{-1}\log\widehat{\mathbb{P}}_n(\Gamma_n) : \Gamma_n \in \mathcal{F}_n, \limsup_{n\to\infty} n^{-1}\log\mathbb{P}_n(\Gamma_n^c) < -\vartheta\Big\},$$

$$\overline{\mathfrak{h}}_\vartheta = \inf\Big\{\limsup_{n\to\infty} n^{-1}\log\widehat{\mathbb{P}}_n(\Gamma_n) : \Gamma_n \in \mathcal{F}_n, \limsup_{n\to\infty} n^{-1}\log\mathbb{P}_n(\Gamma_n^c) < -\vartheta\Big\},$$

$$\mathfrak{h}_\vartheta = \inf\Big\{\lim_{n\to\infty} n^{-1}\log\widehat{\mathbb{P}}_n(\Gamma_n) : \Gamma_n \in \mathcal{F}_n, \limsup_{n\to\infty} n^{-1}\log\mathbb{P}_n(\Gamma_n^c) < -\vartheta\Big\},$$

where the infimum in the last relation is taken over all families $\{\Gamma_n\}$ for which the limit exists.

As functions of ϑ, $\underline{\mathfrak{h}}_\vartheta$, $\overline{\mathfrak{h}}_\vartheta$, and $\mathfrak{h}_\vartheta$ are non-decreasing and non-positive on $[0, +\infty[$, and are equal to $-\infty$ on $]-\infty, 0[$. Moreover, the inequalities $\underline{\mathfrak{h}}_\vartheta \le \overline{\mathfrak{h}}_\vartheta \le \mathfrak{h}_\vartheta \le 0$ hold for $\vartheta \ge 0$.

Proposition 1.7 *Suppose that the FT holds with the rate function (1.13). Then, for $\vartheta \in \mathbb{R}$, we have*

$$\underline{\mathfrak{h}}_\vartheta = \overline{\mathfrak{h}}_\vartheta = \mathfrak{h}_\vartheta = \inf_{\alpha\in[0,1[} \frac{\vartheta\alpha + e(\alpha)}{1-\alpha}.$$

1.5 Interpretation of Theorems A and B

The FT part of Theorem **A** fits directly into the mathematical framework and interpretation of the FT discussed in Section 1.4 with $\Omega_n = M_n$, $\mathbb{P}_n$ defined by (0.2), and $\Theta_n = \theta \circ \varphi^{n-1}$.

The above interpretation does not apply to the setup of Theorem **B**. The FT part of Theorem **B** is related to the *principle of regular entropic fluctuations* of [JPRB11] adapted to the setting of this paper, and provides a *uniformity* counterpart to the FT of Theorem **A** which is both of conceptual and practical (numerical, experimental) importance. Here we shall briefly comment on this point, referring the reader to [CJPS18, CJN$^+$18] for additional discussion.

Let $\mathbb{P}_n$ be as in Theorem **A**. For each fixed $m \ge 0$, since the function σ is bounded, the random variables $(n+m)^{-1}S_{n+m}\sigma$ and $n^{-1}S_n\sigma$ are exponentially equivalent under the law $\mathbb{P}_{n+m}$ as $n \to \infty$, and the theorem implies that for any Borel set $\Gamma \subset \mathbb{R}$,

$$\begin{aligned} -\inf_{s\in\mathring{\Gamma}} I(s) &\le \liminf_{n\to\infty} n^{-1}\log\mathbb{P}_{n+m}\left\{n^{-1}S_n\sigma \in \Gamma\right\} \\ &\le \limsup_{n\to\infty} n^{-1}\log\mathbb{P}_{n+m}\left\{n^{-1}S_n\sigma \in \Gamma\right\} \le -\inf_{s\in\overline{\Gamma}} I(s). \end{aligned} \tag{1.18}$$

If along some subsequence $\mathbb{P}_{m_k} \rightharpoonup \mathbb{P}$, a natural question is whether (1.18) holds with $\mathbb{P}_{n+m}$ replaced with $\mathbb{P}$. Theorem **B** gives a positive answer if $\mathbb{P}$ is a weak Gibbs measure with the same potential G as in the sequence $\mathbb{P}_n$, and further asserts that this is the only requirement for $\mathbb{P}$; neither weak convergence nor invariance of $\mathbb{P}$ play a role. Such level of uniformity is a somewhat surprising strengthening of the principle of regular entropic fluctuations in the setting of chaotic dynamical systems on compact metric spaces.

1.6 Outlook

The general mathematical framework and interpretation of the FT presented in this section are rooted in pioneering works on the subject [ECM93, ES94, GC95b, GC95a, LS99, Mae99]. As formulated here, they were developed in special cases in [JOPS12, BJPP17], and in full generality in [CJN$^+$18].

When additional mathematical/physical structure is available, one can say more. For example, for open stochastic or Hamiltonian systems which carry energy fluxes generated by temperature differentials, the entropy production observable coincides with thermodynamic entropy production; see [JPRB11, JPS17, CJPS18]. For the chaotic dynamical systems (M, φ) considered in this paper, Theorems **A** and **B** show that the FT is a structural feature of the thermodynamic formalism. Relaxing the chaoticity assumptions (expansiveness and specification) brings forward a number of important open problems that remain to be discussed in the future.

2 Preliminaries

2.1 A class of continuous dynamical systems

Let M be a compact metric space with metric d and Borel σ-algebra $\mathcal{B}(M)$. We recall that $C(M)$ (respectively, $B(M)$) denotes the Banach space of continuous (bounded measurable) functions $f : M \to \mathbb{R}$ with the norm $\|f\|_\infty = \sup_{x\in M} |f(x)|$. The set $\mathcal{P}(M)$ of Borel probability measures on M is endowed with the topology of weak convergence (denoted $\rightharpoonup$) and the corresponding Borel σ-algebra. Given $V \in C(M)$ and $\mathbb{Q} \in \mathcal{P}(M)$, we denote by $\langle V, \mathbb{Q}\rangle$ the integral of V with respect to $\mathbb{Q}$. In the following, we shall always assume:

(C) $\varphi : M \to M$ *is a continuous map.*

On occasions, we shall strengthen the above standing assumption to

(H) $\varphi : M \to M$ *is a homeomorphism.*

In the sequel we shall always explicitly mention when Condition **(H)** is assumed. The reversal operation as defined in the introduction makes sense only if **(H)** holds. The transformations that lead to the FR and FT for general continuous maps are defined in Section 3.2.

The set of φ-invariant elements of $\mathcal{P}(M)$ is denoted by $\mathcal{P}_\varphi(M)$, and the set of φ-ergodic measures by $\mathcal{E}_\varphi(M)$. The topological entropy of φ is denoted by $h_{\mathrm{Top}}(\varphi)$. The Kolmogorov–Sinai entropy of φ with respect to $\mathbb{Q} \in \mathcal{P}_\varphi(M)$ is denoted by $h_\varphi(\mathbb{Q})$.

The *orbit* of a point $x \in M$ is defined as $\{\varphi^k(x)\}_{k\in\mathbb{Z}_+}$. An orbit is said to be n*-periodic* if $\varphi^n(x) = x$, and we denote by M_n the set of fixed points of φ^n.

For $I = [\![l, m]\!] \subset \mathbb{Z}_+$, we denote orbit segments by

$$\varphi^I(x) = \{\varphi^k(x)\}_{k\in I},$$

and call *specification* a finite family of such segments

$$\xi = \{\varphi^{I_i}(x_i)\}_{i\in[\![1,n]\!]}.$$

The integers n and $L(\xi) = \max\{|k - k'| : k, k' \in \cup_{i\in[\![1,n]\!]} I_i\}$ are called the *rank* and the *length* of ξ respectively. The specification ξ is N*-separated* whenever $d(I_i, I_j) = \min_{k_i\in I_i, k_j\in I_j} |k_i - k_j| \geq N$ for all distinct $i, j \in [\![1, n]\!]$. It is ϵ*-shadowed* by $x \in M$ whenever

$$\max_{i\in[\![1,n]\!]} \max_{k\in I_i} d(\varphi^k(x), \varphi^k(x_i)) < \epsilon.$$

Given $x \in M$, $n \geq 0$, and $\epsilon > 0$, the *Bowen ball* is defined by

$$B_n(x, \epsilon) = \{y \in M : d(\varphi^k(y), \varphi^k(x)) < \epsilon \text{ for } 0 \leq k \leq n-1\}.$$

Many variants of the specification property appear in the literature; see [KLO16] for a review. We shall make use of the following two forms:

(WPS) *φ has the* weak periodic specification property *if for any $\delta > 0$ there is a sequence of integers $\{m_\delta(n)\}_{n \geq n_0}$ such that $0 \leq m_\delta(n) < n$ for $n \geq n_0$, $\lim_{\delta \downarrow 0} \lim_{n \to \infty} n^{-1} m_\delta(n) = 0$, and for any $x \in M$ and $n \geq n_0$, we have $M_n \cap B_{n - m_\delta(n)}(x, \delta) \neq \varnothing$.*

(S) *φ has the* specification property *if for any $\delta > 0$ there is $N(\delta) \geq 1$ such that any $N(\delta)$-separated specification $\xi = \{\varphi^{I_i}(x_i)\}_{i \in [\![1,n]\!]}$ is δ-shadowed by some $x \in M$.*

Remark 2.1 We shall also refer to *Bowen's specification property* [Bow74] as the property **(S)** with the additional constraint that $x \in M_{L(\xi)+N(\delta)}$. Bowen's specification obviously implies both **(WPS)** and **(S)**.

The weak periodic specification property is well suited for the LDP of Theorem **A**. Together with expansiveness (see Definition 2.6 below), it is also sufficient to justify a large part of the thermodynamic formalism involved in the proof of this result. It is not needed for Theorem **B**.

The specification Property **(S)** is involved in the proof of both Theorems **A** and **B**. However, it is only needed to ensure the conclusion of the following proposition (see Theorem B in [EKW94], whose proof given for case **(H)** extends without change to any continuous φ). If the conclusion of the latter can be obtained by a different argument, then Property **(S)** is not needed at all.

Proposition 2.2 *Suppose that φ satisfies Condition* **(S)**. *Then, the set $\mathcal{E}_\varphi(M)$ of ergodic measures is* entropy-dense *in $\mathcal{P}_\varphi(M)$, i.e., for any $\mathbb{P} \in \mathcal{P}_\varphi(M)$ there exists a sequence $\{\mathbb{P}_m\} \subset \mathcal{E}_\varphi(M)$ such that*

$$\mathbb{P}_m \rightharpoonup \mathbb{P}, \qquad h_\varphi(\mathbb{P}_m) \to h_\varphi(\mathbb{P}) \quad \text{as } m \to \infty.$$

2.2 Asymptotic additivity

In this section we establish two technical results about asymptotically additive potential sequences. The first result is related to [FH10, Proposition A.1 (1) and Lemma A.4]. See also [BV15, Proposition 3.2].

Lemma 2.3 *For all $\mathbb{P} \in \mathcal{P}_\varphi(M)$ and $\mathcal{G} \in \mathcal{A}(M)$, the limit*

$$\mathcal{G}(\mathbb{P}) = \lim_{n \to \infty} n^{-1} \langle G_n, \mathbb{P} \rangle \tag{2.1}$$

exists and is finite. Moreover, for any approximating sequence $\{G^{(k)}\}$ of $\mathcal{G}$ we have

$$\mathcal{G}(\mathbb{P}) = \lim_{k \to \infty} \langle G^{(k)}, \mathbb{P} \rangle. \tag{2.2}$$

The convergence in (2.1) and (2.2) is uniform in $\mathbb{P} \in \mathcal{P}_\varphi(M)$, and the real-valued function $\mathbb{P} \mapsto \mathcal{G}(\mathbb{P})$ is continuous on the space $\mathcal{P}_\varphi(M)$ endowed with the weak topology.

Proof. For all k, m, n and all $\mathbb{P} \in \mathcal{P}_\varphi(M)$ we have

$$\begin{aligned} \left|\frac{1}{n}\langle G_n, \mathbb{P}\rangle - \frac{1}{m}\langle G_m, \mathbb{P}\rangle\right| &\leq \left|\frac{1}{n}\langle G_n, \mathbb{P}\rangle - \langle G^{(k)}, \mathbb{P}\rangle\right| + \left|\frac{1}{m}\langle G_m, \mathbb{P}\rangle - \langle G^{(k)}, \mathbb{P}\rangle\right| \\ &\leq \frac{1}{n}\left\|G_n - S_n G^{(k)}\right\|_\infty + \frac{1}{m}\left\|G_m - S_m G^{(k)}\right\|_\infty . \end{aligned} \tag{2.3}$$

In view of (0.11), we conclude that $\{n^{-1}\langle G_n, \mathbb{P}\rangle\}$ is a Cauchy sequence and hence the limit (2.1) exists and is finite. Letting $m \to \infty$ in (2.3) and using again (0.11), we conclude that the limit in (2.1) is uniform in $\mathbb{P} \in \mathcal{P}_\varphi(M)$. Since

$$\sup_{\mathbb{P}\in\mathcal{P}_\varphi}\left|\frac{1}{n}\langle G_n, \mathbb{P}\rangle - \langle G^{(k)}, \mathbb{P}\rangle\right| \leq \frac{1}{n}\left\|G_n - S_n G^{(k)}\right\|_\infty,$$

the uniform convergence in (2.1) and relation (0.11) give that the convergence in (2.2) is also uniform in $\mathbb{P} \in \mathcal{P}_\varphi(M)$. This last uniform convergence and the continuity of $\mathbb{P} \mapsto \langle G^{(k)}, \mathbb{P}\rangle$ yield the continuity of $\mathbb{P} \mapsto \mathcal{G}(\mathbb{P})$. □

The second result concerns the variations of $\mathcal{G} \in \mathcal{A}(M)$.

Lemma 2.4 *For any $\mathcal{G} = \{G_n\} \in \mathcal{A}(M)$,*[16]

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty}\sup_{x\in M}\sup_{y,z\in B_n(x,\delta)}\frac{1}{n}\left|G_n(y) - G_n(z)\right| = 0. \tag{2.4}$$

Moreover, if $m_\delta(n)$ satisfies $\lim_{\delta\downarrow 0}\limsup_{n\to\infty}\frac{1}{n}m_\delta(n) = 0$, then

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty}\frac{1}{n}\|G_{n-m_\delta(n)} - G_n\|_\infty = 0, \tag{2.5}$$

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty}\sup_{x\in M}\sup_{y,z\in B_{n-m_\delta(n)}(x,\delta)}\frac{1}{n}\left|G_n(y) - G_n(z)\right| = 0. \tag{2.6}$$

Proof. We first prove (2.4). Let $\{G^{(k)}\} \subset C(M)$ be an approximating sequence for $\mathcal{G}$. Let $\epsilon > 0$, and fix k large enough so that $n^{-1}\left\|G_n - S_n(G^{(k)})\right\|_\infty < \epsilon$ for all $n \geq N(k)$. Let $\delta > 0$ be such that $|G^{(k)}(x_1) - G^{(k)}(x_2)| < \epsilon$ for $d(x_1, x_2) < 2\delta$. Then, for all $n \geq N(k)$, $x \in M$, and $y, z \in B_n(x, \delta)$, we have

$$\left|G_n(y) - G_n(z)\right| \leq 2n\epsilon + \left|S_n G^{(k)}(y) - S_n G^{(k)}(z)\right| \leq 3n\epsilon.$$

Since $\epsilon > 0$ is arbitrary, this establishes (2.4).

To prove (2.5), fix $\epsilon > 0$ and k, N large enough so that $\|S_n G^{(k)} - G_n\|_\infty \leq n\epsilon$ for $n \geq N$. Then, for any fixed $\delta > 0$, we have for all n large enough that $n - m_\delta(n) \geq N$, and hence that

$$\begin{aligned} \|G_{n-m} - G_n\|_\infty &\leq \|G_{n-m} - S_{n-m}G^{(k)}\|_\infty + \|S_{n-m}G^{(k)} - S_n G^{(k)}\|_\infty + \|S_n G^{(k)} - G_n\|_\infty \\ &\leq (n-m)\epsilon + \|S_{n-m}G^{(k)} - S_n G^{(k)}\|_\infty + n\epsilon \leq 2n\epsilon + m\,\|G^{(k)}\|_\infty, \end{aligned}$$

[16] We note that (2.4) is proved in [ZZC11, Lemma 2.1].

where $m = m_\delta(n)$. Using the condition on $m_\delta(n)$, this gives

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty}\frac{1}{n}\|G_{n-m_\delta(n)} - G_n\|_\infty \leq 2\epsilon.$$

Since $\epsilon > 0$ is arbitrary, we obtain (2.5).

Finally, to prove (2.6), we observe that for each $x \in M$ and $y, z \in B_{n-m}(x,\delta)$,

$$\big|G_n(y) - G_n(z)\big| \leq 2\|G_{n-m} - G_n\|_\infty + \big|G_{n-m}(y) - G_{n-m}(z)\big|.$$

The required relation (2.6) follows now from (2.4), (2.5), and the condition on $m_\delta(n)$. □

2.3 Topological pressure

We now introduce a notion of *topological pressure* associated with φ which generalizes the usual concept of topological pressure to asymptotically additive potential sequences. Given $\epsilon > 0$ and an integer $n \geq 1$, a finite set $E \subset M$ is called (ϵ, n)*-separated* if $y \notin B_n(x,\epsilon)$ for any distinct $x, y \in E$, and (ϵ, n)*-spanning* if $\{B_n(x,\epsilon)\}_{x\in E}$ covers M. For $\mathcal{G} = \{G_n\} \in \mathcal{A}(M)$, $\epsilon > 0$, and $n \geq 1$ we define

$$S(\mathcal{G},\epsilon,n) = \inf\left\{\sum_{x\in E} e^{G_n(x)} : E \text{ is } (\epsilon,n)\text{-spanning}\right\}, \tag{2.7}$$

$$N(\mathcal{G},\epsilon,n) = \sup\left\{\sum_{x\in E} e^{G_n(x)} : E \text{ is } (\epsilon,n)\text{-separated}\right\}. \tag{2.8}$$

We shall show that

$$\lim_{\epsilon\downarrow 0}\limsup_{n\to\infty}\frac{1}{n}\log S(\mathcal{G},\epsilon,n) = \lim_{\epsilon\downarrow 0}\liminf_{n\to\infty}\frac{1}{n}\log S(\mathcal{G},\epsilon,n), \tag{2.9}$$

$$\lim_{\epsilon\downarrow 0}\limsup_{n\to\infty}\frac{1}{n}\log N(\mathcal{G},\epsilon,n) = \lim_{\epsilon\downarrow 0}\liminf_{n\to\infty}\frac{1}{n}\log N(\mathcal{G},\epsilon,n). \tag{2.10}$$

Moreover, these two quantities coincide and their common value, which we denote by $\mathfrak{p}_\varphi(\mathcal{G})$, will be called the *topological pressure* of φ with respect to $\mathcal{G}$. This result is of course well known in the additive case $\mathcal{G} = \{S_n(G)\}$ when $G \in C(M)$, and we shall write

$$\mathfrak{p}^0_\varphi(G) := \mathfrak{p}_\varphi(\{S_nG\}).$$

Besides existence, we shall also establish some basic properties of $\mathfrak{p}_\varphi$. They will be proven by approximation arguments, starting from the corresponding well-known results in the additive case.

Proposition 2.5 (1) *The relations (2.9) and (2.10) hold and the respective quantities coincide. Moreover, the map*

$$\mathcal{A}(M) \ni \mathcal{G} \mapsto \mathfrak{p}_\varphi(\mathcal{G}) \in]-\infty,+\infty]$$

is convex, and either $\mathfrak{p}_\varphi(\mathcal{G}) = +\infty$ *for all* $\mathcal{G} \in \mathcal{A}(M)$*, or* $\mathfrak{p}_\varphi(\mathcal{G}) \in \mathbb{R}$ *for all* $\mathcal{G} \in \mathcal{A}(M)$.

(2) *If* $\{G^{(k)}\}$ *is an approximating sequence for* $\mathcal{G}$*, then*

$$\mathfrak{p}_\varphi(\mathcal{G}) = \lim_{k\to\infty}\mathfrak{p}^0_\varphi(G^{(k)}). \tag{2.11}$$

(3) *The topological entropy of* φ *satisfies*

$$h_{\rm Top}(\varphi) = \mathfrak{p}^0_\varphi(0) = \sup_{\mathbb{Q}\in\mathcal{P}_\varphi(M)} h_\varphi(\mathbb{Q}) = \sup_{\mathbb{Q}\in\mathcal{E}_\varphi(M)} h_\varphi(\mathbb{Q}). \tag{2.12}$$

In particular, $\mathfrak{p}_\varphi(\mathcal{G})$ *is finite for all* $\mathcal{G}\in\mathcal{A}(M)$ *if and only if* φ *has finite topological entropy.*

(4) *If the topological entropy of* φ *is finite, then for any* $\mathcal{G},\mathcal{G}'\in\mathcal{A}(M)$ *we have*

$$|\mathfrak{p}_\varphi(\mathcal{G}) - \mathfrak{p}_\varphi(\mathcal{G}')| \le \|\mathcal{G}-\mathcal{G}'\|_*. \tag{2.13}$$

(5) *For any* $\mathcal{G}\in\mathcal{A}(M)$ *we have*

$$\mathfrak{p}_\varphi(\mathcal{G}) = \sup_{\mathbb{P}\in\mathcal{P}_\varphi(M)} \big(\mathcal{G}(\mathbb{P}) + h_\varphi(\mathbb{P})\big). \tag{2.14}$$

(6) *If the entropy map* $\mathcal{P}_\varphi(M)\ni\mathbb{P}\mapsto h_\varphi(\mathbb{P})$ *is upper semicontinuous, then for any* $\mathbb{P}\in\mathcal{P}_\varphi(M)$*,*

$$h_\varphi(\mathbb{P}) = \inf_{\mathcal{G}\in\mathcal{A}(M)} (\mathfrak{p}_\varphi(\mathcal{G}) - \mathcal{G}(\mathbb{P})), \tag{2.15}$$

and for any $\mathcal{G}\in\mathcal{A}(M)$ *and* $\mathbb{P}\in\mathcal{P}_\varphi(M)$ *we have*

$$h_\varphi(\mathbb{P}) = \inf_{G\in C(M)} (\mathfrak{p}_\varphi(\mathcal{G}_G) - \mathcal{G}_G(\mathbb{P})), \tag{2.16}$$

where $\mathcal{G}_G = \{G_n + S_nG\}$*.*

Proof. *Proof of (1) and (2).* As we have already mentioned, if $\mathcal{G}=\{S_nG\}$ for some $G\in C(M)$, then it is well known (see Sections 3.1.b and 20.2 of [KH95] or Section 9.1 of [Wal82]) that the four quantities in (2.9) and (2.10) are equal and define $\mathfrak{p}^0_\varphi(G)$. To extend this result to any $\mathcal{G}=\{G_n\}\in\mathcal{A}(M)$, we start with (2.9). Note that for any finite set $E\subset M$ and any $\mathcal{G}'=\{G'_n\}\in\mathcal{A}(M)$ we have

$$\left|\frac{1}{n}\log\sum_{x\in E} e^{G_n(x)} - \frac{1}{n}\log\sum_{x\in E} e^{G'_n(x)}\right| \le \frac{1}{n}\big\|G_n - G'_n\big\|_\infty. \tag{2.17}$$

It follows that

$$\frac{1}{n}\big|\log S(\mathcal{G},\epsilon,n) - \log S(\mathcal{G}',\epsilon,n)\big| \le \frac{1}{n}\big\|G_n - G'_n\big\|_\infty. \tag{2.18}$$

Taking $\mathcal{G}'=\{S_nG^{(k)}\}$ for some approximating sequence $\{G^{(k)}\}$ of $\mathcal{G}$, we find

$$\frac{1}{n}\left|\log S(\mathcal{G},\epsilon,n) - \log S(\{S_nG^{(k)}\},\epsilon,n)\right| \le \frac{1}{n}\big\|G_n - S_n(G^{(k)})\big\|_\infty.$$

Relation (0.11) and the fact that $\mathfrak{p}^0_\varphi(G^{(k)})$ is well defined for all k give that (2.9) holds and that the limits are equal to $\lim_{k\to\infty}\mathfrak{p}^0_\varphi(G^{(k)})$. Relation (2.17) gives that (2.18) also holds for $\log N(\cdot,\epsilon,n)$, and the above argument yields that the all four limits in (2.9) and (2.10) are equal and that (2.11) holds.[17]

Since $G_n\mapsto\log\sum_{x\in E}e^{G_n(x)}$ is convex by Hölder's inequality, we obtain that so is $\mathcal{G}\mapsto\mathfrak{p}_\varphi(\mathcal{G})$.

[17] See also [IY17, Lemma 1.1.] and [ZZC11, Lemma 2.3] for proofs of (2.11).

By applying (0.13) to the asymptotically additive potential $\mathcal{G} - \mathcal{G}'$, we obtain that the right-hand side of (2.18) is bounded uniformly in n. As a consequence, we have $\mathfrak{p}_\varphi(\mathcal{G}) = \infty \iff \mathfrak{p}_\varphi(\mathcal{G}') = \infty$, which yields the last statement of Part (1).

Proof of (3) and (4). Relations (2.12) are well known (see Theorem 8.6 and Theorem 9.7 (i) in [Wal82]), and (2.13) immediately follows from (2.18).

Proof of (5). The variational principle (2.14) is established in [FH10, Theorem 3.1] for asymptotically *sub*-additive potential sequences. We include here a proof in the (simpler) asymptotically additive case (see also [Bar11, Theorem 7.2.1]).

If $h_\varphi(\mathbb{P}) = +\infty$ for some $\mathbb{P} \in \mathcal{P}_\varphi(M)$, then $\mathfrak{p}_\varphi^0(0) = +\infty$, so that $\mathfrak{p}_\varphi(\mathcal{G}) = +\infty$ for all $\mathcal{G} \in \mathcal{A}(M)$. Relation (2.14) is obvious in this case. Assume now that $h_\varphi(\mathbb{P}) < +\infty$ for all $\mathbb{P} \in \mathcal{P}_\varphi(M)$. By [Wal82, Theorem 9.10], for any $G \in C(M)$,

$$\mathfrak{p}_\varphi^0(G) = \sup_{\mathbb{P}\in\mathcal{P}_\varphi(M)} \left(h_\varphi(\mathbb{P}) + \langle G, \mathbb{P}\rangle\right). \tag{2.19}$$

By Lemma 2.3, the sequence $f_k(\mathbb{P}) := \langle G^{(k)}, \mathbb{P}\rangle + h_\varphi(\mathbb{P})$ converges uniformly to $\mathcal{G}(\mathbb{P}) + h_\varphi(\mathbb{P})$ on $\mathcal{P}_\varphi(M)$. It follows that

$$\begin{aligned}\mathfrak{p}_\varphi(\mathcal{G}) &= \lim_{k\to\infty} \mathfrak{p}_\varphi^0(G^{(k)}) = \lim_{k\to\infty} \sup_{\mathbb{P}\in\mathcal{P}_\varphi(M)} f_k(\mathbb{P}) \\ &= \sup_{\mathbb{P}\in\mathcal{P}_\varphi(M)} \lim_{k\to\infty} f_k(\mathbb{P}) = \sup_{\mathbb{P}\in\mathcal{P}_\varphi(M)} \big(\mathcal{G}(\mathbb{P}) + h_\varphi(\mathbb{P})\big),\end{aligned}$$

where the second equality uses (2.19).

Proof of (6). We start by recalling (see [Wal82, Theorem 9.12]) that if $\mathcal{P}_\varphi(M) \ni \mathbb{P} \mapsto h_\varphi(\mathbb{P})$ is upper semicontinuous, then for all $\mathbb{P} \in \mathcal{P}_\varphi$,

$$h_\varphi(\mathbb{P}) = \inf_{G\in C(M)} (\mathfrak{p}_\varphi^0(G) - \langle G, \mathbb{P}\rangle). \tag{2.20}$$

We now prove (2.15) and (2.16). Both "$\le$" inequalities are an immediate consequence of (2.14).

The "$\ge$" inequality in (2.15) is immediate by (2.20) since for $G \in C(M)$ and $\mathcal{G} = \{S_n G\}$ we have $\mathfrak{p}_\varphi(\mathcal{G}) - \mathcal{G}(\mathbb{P}) = \mathfrak{p}_\varphi^0(G) - \langle G, \mathbb{P}\rangle$. To prove the "$\ge$" inequality in (2.16), fix $\epsilon > 0$ and use (2.20) to find $W_\epsilon \in C(M)$ such that

$$h_\varphi(\mathbb{Q}) \ge \mathfrak{p}_\varphi^0(W_\epsilon) - \langle W_\epsilon, \mathbb{Q}\rangle - \epsilon. \tag{2.21}$$

Consider the sequence $G_k = W_\epsilon - G^{(k)}$, where $\{G^{(k)}\}$ is an approximating sequence for $\mathcal{G}$. In view of (2.2) and (2.13), we have

$$\mathcal{G}_{G_k}(\mathbb{Q}) \to \langle W_\epsilon, \mathbb{Q}\rangle, \quad \mathfrak{p}_\varphi(\mathcal{G}_{G_k}) \to \mathfrak{p}_\varphi^0(W_\epsilon) \quad \text{as } k \to \infty.$$

Combining this with (2.21), we see that for a sufficiently large k,

$$\mathcal{G}_{G_k}(\mathbb{Q}) - \mathfrak{p}_\varphi(\mathcal{G}_{V_k}) \ge \langle W_\epsilon, \mathbb{Q}\rangle - \mathfrak{p}_\varphi^0(W_\epsilon) - \epsilon \ge -h_\varphi(\mathbb{Q}) - 2\epsilon.$$

Since $\epsilon > 0$ is arbitrary, the proof is complete. □

2.4 Expansiveness

Besides the specification properties **(WPS)** and **(S)**, we shall need the following assumptions to formulate our main results. The first concerns the regularity of the entropy map and is needed in the proof of both Theorems **A** and **B**. The second concerns the approximation of the pressure in terms of periodic orbits and is only needed in the proof of Theorem **A**.

(USCE) *φ has finite topological entropy, and the map $\mathbb{Q} \in \mathcal{P}_\varphi(M) \mapsto h_\varphi(\mathbb{Q})$ is upper semicontinuous.*

(PAP) *For any $n \geq 1$ the set M_n is finite, and for any $\mathcal{G} \in \mathcal{A}(M)$, the topological pressure satisfies*

$$\mathfrak{p}_\varphi(\mathcal{G}) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{x \in M_n} \mathrm{e}^{G_n(x)}. \tag{2.22}$$

We note that, by Proposition 2.5, Condition **(USCE)** implies that the pressure $\mathfrak{p}_\varphi(\mathcal{G})$ is finite for all $\mathcal{G} \in \mathcal{A}(M)$ and that the entropy satisfies the variational principle (2.15). In particular, $h_\varphi(\mathbb{Q}) \in [0, \infty[$ for all $\mathbb{Q} \in \mathcal{P}_\varphi$.

We now discuss some criteria ensuring Conditions **(USCE)** and **(PAP)**. Together with the specification property, expansiveness is often considered as characteristic of chaotic dynamics.

Definition 2.6 *The map φ is said to be* forward expansive *if there is $r > 0$ such that if $x, y \in M$ satisfy the inequality $d(\varphi^k(x), \varphi^k(y)) \leq r$ for all $k \in \mathbb{Z}_+$, then $x = y$. A homeomorphism φ is called* expansive *if there is $r > 0$ such that if $x, y \in M$ satisfy the inequality $d(\varphi^k(x), \varphi^k(y)) \leq r$ for all $k \in \mathbb{Z}$, then $x = y$.*

The number r for which this property hold is called the expansiveness (or expansivity) constant of φ. We note that the expansiveness constant depends on the metric d, whereas expansiveness only depends on the induced topology on M.

Theorem 2.7 (1) *If φ is forward expansive or expansive, then Condition* **(USCE)** *holds.*

(2) *If, in addition, φ satisfies Condition* **(WPS)***, then Condition* **(PAP)** *holds.*

Proof. Part (1) follows from Corollary 7.11.1 in [Wal82], whose proof immediately extends from case **(H)** to case **(C)** (see also [Bar11, Lemma 2.4.4]).

Part (2) is also well known in the additive case, and the proof requires only notational modifications in the asymptotically additive case. We include the proof for completeness.

Let $\mathcal{I} = \mathbb{Z}_+$ in the forward expansive case, and $\mathcal{I} = \mathbb{Z}$ in the expansive case. The uniform continuity of φ and periodicity imply that there is $\delta > 0$ such that if $x, y \in M_n$ and $d(x, y) \leq \delta$, then $d(\varphi^k(x), \varphi^k(y)) \leq r$ for all $k \in \mathcal{I}$, where r is the expansivity constant. By expansiveness, such points must coincide, and compactness implies that M_n may contain only finitely many points.

Let $\mathcal{G} \in \mathcal{A}(M)$, and let $\epsilon \in]0, r[$. Then, in view of periodicity, for any $n \geq 1$ the set M_n is (ϵ, n)-separated. It follows that

$$\sum_{x \in M_n} \mathrm{e}^{G_n(x)} \leq N(\mathcal{G}, \epsilon, n),$$

whence, recalling representation (2.10) for the pressure, we conclude that

$$\limsup_{n\to\infty} \frac{1}{n} \log \sum_{x \in M_n} \mathrm{e}^{G_n(x)} \leq \mathfrak{p}_\varphi(\mathcal{G}). \tag{2.23}$$

On the other hand, since φ satisfies Condition **(WPS)**, the set M_n is $(\epsilon, n - m_\epsilon(n))$-spanning. It follows that

$$\sum_{x \in M_n} \mathrm{e}^{G_n(x)} \geq S(\mathcal{G}, \epsilon, n - m_\epsilon(n)).$$

Combining this with representation (2.9), and using the condition on $m_\epsilon(n)$, we see that

$$\liminf_{n\to\infty} \frac{1}{n} \log \sum_{x \in M_n} \mathrm{e}^{G_n(x)} \geq \mathfrak{p}_\varphi(\mathcal{G}). \tag{2.24}$$

Inequalities (2.23) and (2.24) imply the required relation (2.22). □

Recall the variational principle (2.14). A measure $\mathbb{P} \in \mathcal{P}_\varphi(M)$ is called an *equilibrium state* for $\mathcal{G} \in \mathcal{A}(M)$ if

$$\mathfrak{p}_\varphi(\mathcal{G}) = \mathcal{G}(\mathbb{P}) + h_\varphi(\mathbb{P}). \tag{2.25}$$

We refer the reader to Section 9.5 in [Wal82] for a discussion of equilibrium states. The following proposition implies, in particular, that there always exists at least one equilibrium state if **(USCE)** and **(PAP)** are assumed.

Proposition 2.8 *Assume that Conditions* **(USCE)** *and* **(PAP)** *hold, and let*

$$\overline{\mathbb{P}}_n = \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{P}_n \circ \varphi^{-i},$$

where $\mathbb{P}_n$ is defined by (0.14). If $\mathbb{P}$ is a weak limit point of the sequence $\{\overline{\mathbb{P}}_n\}$, then $\mathbb{P}$ is an equilibrium state for $\mathcal{G}$. In particular, if $\mathcal{G}$ is additive, then $\overline{\mathbb{P}}_n = \mathbb{P}_n$, and any limit point of $\{\mathbb{P}_n\}$ is an equilibrium state.

Proof. Let a sequence $n_k \to \infty$ be such that $\overline{\mathbb{P}}_{n_k} \rightharpoonup \mathbb{P}$. Since $\overline{\mathbb{P}}_n$ is invariant for all n, so is $\mathbb{P}$. In view of Hölder's inequality, the function $\mathcal{G} \mapsto \log Z_n(\mathcal{G}) \in \mathbb{R}$ is convex. Moreover, for any $\mathcal{G}, \mathcal{G}' \in \mathcal{A}(M)$, the function $f(\alpha) := \log Z_n(\mathcal{G} + \alpha\mathcal{G}')$ differentiable, and its derivative at zero is given by $f'(0) = \langle G_n', \mathbb{P}_n \rangle$. By convexity, we have $f(1) - f(0) \geq f'(0)$, which gives

$$\log Z_n(\mathcal{G} + \mathcal{G}') - \log Z_n(\mathcal{G}) \geq \langle G_n', \mathbb{P}_n \rangle.$$

Replacing $\mathcal{G} = \{G_n\}$ and $\mathcal{G}' = \{G_n'\}$ with $\{G_n \circ \varphi^i\}$ and $\{G_n' \circ \varphi^i\}$, respectively, using the relation $Z_n(\mathcal{G}) = Z_n(\{G_n \circ \varphi^i\})$, and averaging with respect to n, we derive

$$\log Z_n(\mathcal{G} + \mathcal{G}') - \log Z_n(\mathcal{G}) \geq \langle G_n', \overline{\mathbb{P}}_n \rangle.$$

We take $n = n_k$, divide the above inequality by n_k, and pass to the limit as $k \to \infty$. By (2.1) and the uniformity of the limit on invariant measures, the right-hand side converges to $\mathcal{G}'(\mathbb{P})$, and in view of (2.22), the left-hand side converges to $\mathfrak{p}_\varphi(\mathcal{G} + \mathcal{G}') - \mathfrak{p}_\varphi(\mathcal{G})$. This leads to the inequality $\mathfrak{p}_\varphi(\mathcal{G} + \mathcal{G}') - \mathfrak{p}_\varphi(\mathcal{G}) \geq \mathcal{G}'(\mathbb{P})$, which can be rewritten as

$$\mathfrak{p}_\varphi(\mathcal{G} + \mathcal{G}') - (\mathcal{G} + \mathcal{G}')(\mathbb{P}) \geq \mathfrak{p}_\varphi(\mathcal{G}) - \mathcal{G}(\mathbb{P}).$$

Taking the supremum over $\mathcal{G}' \in \mathcal{A}(M)$ and invoking (2.15), we arrive at (2.25). The statement about the case where $\mathcal{G}$ is additive is immediate, since then $\mathbb{P}_n$ is invariant. □

Remark 2.9 None of the quantities appearing in (2.25) depend on the specific choice of potential $\mathcal{G}$ within a given equivalence class (in the sense of Remark 0.11), and hence the equilibrium states depend only on the equivalence class. The limit points of $\{\overline{\mathbb{P}}_n\}$, however, may depend on the specific choice of $\mathcal{G}$ in the equivalence class. It is an interesting question to describe potentials $\mathcal{G} \in \mathcal{A}(M)$ for which the invariant weak limit points of $\{\mathbb{P}_n\}$ are equilibrium states.

3 Periodic Orbits Fluctuation Principle

3.1 LDP for empirical measures

Let $\mathcal{G} \in \mathcal{A}(M)$. Recall that the sequence of probability measures $\{\mathbb{P}_n\}$ is defined by (0.14), and the sequence of empirical measures $\{\mu_n^x\}$ by (0.4). For a fixed $n \geq 1$, we regard μ_n^x as a random variable on M with range in the space of probability measures $\mathcal{P}(M)$ endowed with the weak topology.

Theorem 3.1 *Suppose that Conditions* **(USCE)**, **(WPS)**, **(S)** *and* **(PAP)** *hold. Then:*

(1) *The LDP holds for $\{\mu_n^{\cdot}\}$ under the laws $\mathbb{P}_n$, with the lower semicontinuous convex rate function $\mathbb{I} : \mathcal{P}(M) \to [0, +\infty]$ defined by*

$$\mathbb{I}(\mathbb{Q}) = \begin{cases} -\mathcal{G}(\mathbb{Q}) - h_\varphi(\mathbb{Q}) + \mathfrak{p}_\varphi(\mathcal{G}) & \text{for } \mathbb{Q} \in \mathcal{P}_\varphi(M), \\ +\infty & \text{otherwise.} \end{cases} \tag{3.1}$$

In other words, for any Borel subset $\Gamma \subset \mathcal{P}(M)$, we have

$$\begin{aligned} -\inf_{\mathbb{Q} \in \dot{\Gamma}} \mathbb{I}(\mathbb{Q}) &\leq \liminf_{n\to\infty} n^{-1} \log \mathbb{P}_n\{\mu_n^{\cdot} \in \Gamma\} \\ &\leq \limsup_{n\to\infty} n^{-1} \log \mathbb{P}_n\{\mu_n^{\cdot} \in \Gamma\} \leq -\inf_{\mathbb{Q} \in \overline{\Gamma}} \mathbb{I}(\mathbb{Q}), \end{aligned} \tag{3.2}$$

where $\dot{\Gamma}$ and $\overline{\Gamma}$ stand, respectively, for the interior and closure of Γ.

(2) *For any $\mathcal{V} = \{V_n\} \in \mathcal{A}(M)$, the sequence $\frac{1}{n} V_n$ under the laws $\mathbb{P}_n$ satisfies the LDP with the good convex rate function $I : \mathbb{R} \to [0, +\infty]$ defined by the contraction relation*[18]

$$I(s) = \inf\{\mathbb{I}(\mathbb{Q}) : \mathbb{Q} \in \mathcal{P}_\varphi(M), \mathcal{V}(\mathbb{Q}) = s\}. \tag{3.3}$$

In other words, for any Borel subset $\Gamma \subset \mathbb{R}$, we have

$$\begin{aligned} -\inf_{s \in \dot{\Gamma}} I(s) &\leq \liminf_{n\to\infty} n^{-1} \log \mathbb{P}_n \left\{ n^{-1} V_n \in \Gamma \right\} \\ &\leq \limsup_{n\to\infty} n^{-1} \log \mathbb{P}_n\{n^{-1} V_n \in \Gamma\} \leq -\inf_{s \in \overline{\Gamma}} I(s). \end{aligned} \tag{3.4}$$

Moreover, I is the Legendre transform of the function $\alpha \mapsto \mathfrak{p}_\varphi(\mathcal{G} + \alpha\mathcal{V}) - \mathfrak{p}_\varphi(\mathcal{G})$.

Remark 3.2 An immediate consequence of (3.1) is that the function $\mathbb{I}$ is affine on $\mathcal{P}_\varphi(M)$.

[18] Recall that for $\mathcal{V} \in \mathcal{A}(M)$, the quantity $\mathcal{V}(\mathbb{Q})$ is defined as (2.1).

Remark 3.3 If $\mathcal{V} = \{S_n V\}$ for some $V \in C(M)$, then Part (2) follows from Part (1) by an application of the contraction principle [DZ00, Theorem 4.2.1]. When $\mathcal{V}$ is only asymptotically additive, an approximation argument is required (see Section 5.3). This applies, in particular, to the entropy production sequence $\{\sigma_n\}$ defined by (0.16), see below.

Obviously, Part (1) of Theorem 3.1 yields the LDP part of Theorem **A** in the asymptotically additive setting.

The proof of Theorem 3.1 is postponed to Section 5, more precisely to Theorem 5.2. There some more general measures $\mathbb{P}_n$ are considered, in order to give a unified treatment to the measures $\mathbb{P}_n$ in (0.14) and the weak Gibbs measures considered in Section 4.

3.2 Fluctuation Theorem and Fluctuation Relations

In this subsection we prove the FT and FR parts of Theorem **A** in the general setting of the previous subsection. To this end, in addition to Conditions **(USCE)**, **(WPS)**, **(S)** and **(PAP)** which are needed for the LDP, we impose one of the following assumptions to ensure the validity of FR.

($\mathcal{C}$-Commutation) *There is a homeomorphism $\theta : M \to M$ such that $\theta \circ \theta = \mathrm{Id}_M$ and $\varphi = \theta \circ \varphi \circ \theta$.*

($\mathcal{R}$-Reversal) *The map φ is a homeomorphism and there is a homeomorphism $\theta : M \to M$ such that $\theta \circ \theta = \mathrm{Id}_M$ and $\varphi^{-1} = \theta \circ \varphi \circ \theta$.*

Let us remark that in both cases, if $\mathbb{Q} \in \mathcal{P}_\varphi(M)$, then $\widehat{\mathbb{Q}} := \mathbb{Q} \circ \theta \in \mathcal{P}_\varphi(M)$. Indeed, in case $(\mathcal{R})$, for any $V \in C(M)$ we have

$$\langle V \circ \varphi, \widehat{\mathbb{Q}} \rangle = \langle V \circ \varphi \circ \theta, \mathbb{Q} \rangle = \langle V \circ \theta \circ \varphi^{-1}, \mathbb{Q} \rangle = \langle V \circ \theta, \mathbb{Q} \rangle = \langle V, \widehat{\mathbb{Q}} \rangle.$$

A similar argument applies in case $(\mathcal{C})$.

Condition $(\mathcal{R})$ is the standard dynamical system reversal condition that appears in virtually all works on the FT and FR. To the best of our knowledge, it was not previously remarked that $(\mathcal{C})$ also suffices to derive the FR. Since $(\mathcal{C})$ does *not* require that φ be a homeomorphism, it allows one to expand the class of examples for which the FR can be established.

Example 3.4 Let X be a compact metric space and $\phi : X \to X$ a continuous map. Set $M = X \times X$ and $\theta(x, y) = (y, x)$. Then $(\mathcal{C})$ holds for the map $\varphi : (x, y) \mapsto (\phi(x), \phi(y))$. If ϕ is a homeomorphism, then $(\mathcal{R})$ holds for the homeomorphism $\varphi : (x, y) \mapsto (\phi(x), \phi^{-1}(y))$.

Example 3.5 An interesting concrete setting of Example 3.4 is the case $X = [0, 1]$. The classical examples of interval maps $\phi : [0, 1] \to [0, 1]$ such as the tent map, Farey map, or the Pomeau-Manneville map are not bijections and $(\mathcal{R})$ cannot hold whereas $(\mathcal{C})$ applies. We refer the reader to [CJPS18] for a discussion of the FT and FR for this class of examples.

Observe that Condition $(\mathcal{R})/(\mathcal{C})$ implies

$$S_n(V \circ \theta)(x) = \sum_{0 \le k < n} V \circ \theta \circ \varphi^k(x) = \sum_{0 \le k < n} V \circ \varphi^{\mp k} \circ \theta(x) = (S_n V)(\theta_n^{\mp}(x)) \tag{3.5}$$

where

$$\theta_n^- = \theta \circ \varphi^{n-1} \text{ for } (\mathcal{R}), \qquad \theta_n^+ = \theta \text{ for } (\mathcal{C}).$$

Note that the map $\theta_n^\pm$ is an involution: in the case $(\mathcal{C})$ this is immediate, and in the case $(\mathcal{R})$ we have

$$(\theta_n^-)^{-1} = \varphi^{1-n} \circ \theta^{-1} = \varphi^{1-n} \circ \theta = \theta \circ \varphi^{n-1} = \theta_n^-.$$

For $\mathcal{G} = \{G_n\} \in \mathcal{A}(M)$, we define $\mathcal{G} \circ \theta := \{G_n \circ \theta_n^\pm\}$.

Lemma 3.6 *If $\mathcal{G}$ is asymptotically additive with approximating sequence $\{G^{(k)}\}$, then $\mathcal{G} \circ \theta$ is asymptotically additive with approximating sequence $\{G^{(k)} \circ \theta\}$.*

Proof. By (3.5), we have

$$\left\| G_n \circ \theta_n^\pm - S_n(G^{(k)} \circ \theta) \right\|_\infty = \left\| G_n \circ \theta_n^\pm - S_n(G^{(k)}) \circ \theta_n^\pm \right\|_\infty = \left\| G_n - S_n G^{(k)} \right\|_\infty,$$

and the result follows. □

It follows from Lemmas 2.3 and 3.6 that

$$(\mathcal{G} \circ \theta)(\mathbb{Q}) = \lim_{k\to\infty} \langle G^{(k)} \circ \theta, \mathbb{Q} \rangle = \lim_{k\to\infty} \langle G^{(k)}, \widehat{\mathbb{Q}} \rangle = \mathcal{G}(\widehat{\mathbb{Q}}), \quad \mathbb{Q} \in \mathbb{P}_\varphi(M). \tag{3.6}$$

In addition, we have

Lemma 3.7 *The following holds:*

$$h_\varphi(\widehat{\mathbb{Q}}) = h_\varphi(\mathbb{Q}) \quad \textit{for } \mathbb{Q} \in \mathbb{P}_\varphi(M), \tag{3.7}$$

$$\mathfrak{p}_\varphi(\mathcal{G} \circ \theta) = \mathfrak{p}_\varphi(\mathcal{G}) \quad \textit{for } \mathcal{G} \in \mathcal{A}(M). \tag{3.8}$$

Proof. Let $\psi = \theta \circ \varphi \circ \theta$. Since the Kolmogorov–Sinai entropy is conjugacy invariant (see [Wal82, Theorem 4.11]), $h_\psi(\mathbb{Q}) = h_\varphi(\mathbb{Q} \circ \theta) = h_\varphi(\widehat{\mathbb{Q}})$. In case $(\mathcal{C})$ we have $\psi = \varphi$, and (3.7) follows. In case $(\mathcal{R})$ we note that by [Wal82, Theorem 4.13], $h_\psi(\mathbb{Q}) = h_{\varphi^{-1}}(\mathbb{Q}) = h_\varphi(\mathbb{Q})$. Thus (3.7) holds in both cases.

In order to prove (3.8), we observe that, by (2.14), (3.6) and (3.7),

$$\begin{aligned} \mathfrak{p}_\varphi(\mathcal{G} \circ \theta) &= \sup_{\mathbb{Q} \in \mathcal{P}_\varphi(M)} \big((\mathcal{G} \circ \theta)(\mathbb{Q}) + h_\varphi(\mathbb{Q}) \big) = \sup_{\mathbb{Q} \in \mathcal{P}_\varphi(M)} \big(\mathcal{G}(\widehat{\mathbb{Q}}) + h_\varphi(\widehat{\mathbb{Q}}) \big) \\ &= \sup_{\mathbb{Q} \in \mathcal{P}_\varphi(M)} \big(\mathcal{G}(\mathbb{Q}) + h_\varphi(\mathbb{Q}) \big) = \mathfrak{p}_\varphi(\mathcal{G}), \end{aligned}$$

which completes the proof. □

The *entropy production in time n* is defined by

$$\sigma_n = \sigma_n(\mathcal{G}) = G_n - G_n \circ \theta_n^\pm. \tag{3.9}$$

Observing that M_n is strictly invariant under $\theta_n^\pm$, the following result is immediate.

Lemma 3.8 *Let $\mathcal{G} \in \mathcal{A}(M)$. Then $Z_n(\mathcal{G}) = Z_n(\mathcal{G} \circ \theta) =: Z_n$. Moreover, $\mathbb{P}_n \circ \theta_n^{\pm}$ is the measure given by* (0.14) *for the potential sequence $\mathcal{G} \circ \theta$, i.e.,*

$$\mathbb{P}_n \circ \theta_n^{\pm} = Z_n^{-1} \sum_{x \in M_n} \mathrm{e}^{G_n \circ \theta_n^{\pm}(x)} \delta_x.$$

Finally, the measures $\mathbb{P}_n$ and $\mathbb{P}_n \circ \theta_n^{\pm}$ are equivalent, and we have

$$\log \frac{\mathrm{d}\mathbb{P}_n}{\mathrm{d}\mathbb{P}_n \circ \theta_n^{\pm}} = \sigma_n.$$

Condition **(PAP)** implies that the limit defining the entropic pressure

$$e(\alpha) = \lim_{n\to\infty} \frac{1}{n} \log \int_{M_n} \mathrm{e}^{-\alpha\sigma_n} \mathrm{d}\mathbb{P}_n$$

exists for all $\alpha \in \mathbb{R}$ and is given by

$$e(\alpha) = \mathfrak{p}_{\varphi}((1-\alpha)\mathcal{G} + \alpha\mathcal{G} \circ \theta).$$

For $\mathbb{Q} \in \mathcal{P}_{\varphi}(M)$, let

$$\mathrm{ep}(\mathbb{Q}) = \mathcal{G}(\mathbb{Q}) - (\mathcal{G} \circ \theta)(\mathbb{Q}). \tag{3.10}$$

Note that

$$\mathrm{ep}(\widehat{\mathbb{Q}}) = -\mathrm{ep}(\mathbb{Q}), \tag{3.11}$$

and that if $G_n = S_n V$ and $\mathbb{Q} \in \mathcal{P}_{\varphi}(M)$, then $\mathrm{ep}(\mathbb{Q}) = \langle V - V\circ\theta, \mathbb{Q}\rangle = \langle V, \mathbb{Q} - \widehat{\mathbb{Q}}\rangle$.

Theorem 3.9 *In addition to the hypotheses of Theorem 3.1, suppose that Condition $(\mathcal{C})/(\mathcal{R})$ holds. Then the rate function $\mathbb{I}$ of the LDP for the empirical measures* (0.4) *under the laws $\mathbb{P}_n$ satisfies the relation*[19]

$$\mathbb{I}(\widehat{\mathbb{Q}}) = \mathbb{I}(\mathbb{Q}) + \mathrm{ep}(\mathbb{Q}) \quad \textit{for any } \mathbb{Q} \in \mathcal{P}_{\varphi}(M). \tag{3.12}$$

Furthermore, the sequence $\frac{1}{n}\sigma_n$ under the laws $\mathbb{P}_n$ satisfies the LDP (3.4) *with the good convex rate function given by*

$$I(s) = \inf\{\mathbb{I}(\mathbb{Q}) : \mathbb{Q} \in \mathcal{P}_{\varphi}(M), \mathrm{ep}(\mathbb{Q}) = s\}. \tag{3.13}$$

The rate function I satisfies the relation

$$I(-s) = I(s) + s \tag{3.14}$$

for all $s \in \mathbb{R}$ and is the Legendre transform of $e(-\alpha)$.

Proof. Recall that the rate function $\mathbb{I}$ is given by (3.1). By (3.6) and (3.7), for any $\mathbb{Q} \in \mathcal{P}_{\varphi}(M)$

$$\mathbb{I}(\widehat{\mathbb{Q}}) = -\mathcal{G}(\widehat{\mathbb{Q}}) - h_{\varphi}(\widehat{\mathbb{Q}}) + \mathfrak{p}_{\varphi}(\mathcal{G}) = -(\mathcal{G}\circ\theta)(\mathbb{Q}) - h_{\varphi}(\mathbb{Q}) + \mathfrak{p}_{\varphi}(\mathcal{G}),$$

and (3.12) follows.

[19]Note that (3.12) implies (0.6) for $\mathbb{Q} \in \mathcal{P}_{\varphi}(M)$. On the other hand, when $\mathbb{Q}$ is not invariant, both sides of (0.6) are $+\infty$, whereas in the asymptotically additive setup, the quantity $\mathrm{ep}(\mathbb{Q})$ is defined only for $\mathbb{Q} \in \mathcal{P}_{\varphi}(M)$.

To obtain the LDP for $n^{-1}\sigma_n$, we first observe that, by Lemma 3.6, the sequence $\{\sigma_n\}$ is asymptotically additive with approximating sequence $\sigma^{(k)} = G^{(k)} - G^{(k)} \circ \theta$. The LDP with rate function (3.13) then follows from Part (2) of Theorem 3.1.

Finally, the FR (3.14) follows from Proposition 1.4, but it also can be directly deduced as follows. Combining (3.11) with (3.12) and (3.13), we see that

$$\begin{aligned} I(-s) &= \inf\{\mathbb{I}(\mathbb{Q}) : \mathbb{Q} \in \mathcal{P}_\varphi(M), \mathrm{ep}(\mathbb{Q}) = -s\} \\ &= \inf\{\mathbb{I}(\widehat{\mathbb{Q}}) - \mathrm{ep}(\mathbb{Q}) : \mathbb{Q} \in \mathcal{P}_\varphi(M), \mathrm{ep}(\mathbb{Q}) = -s\} \\ &= \inf\{\mathbb{I}(\mathbb{Q}') : \mathbb{Q}' \in \mathcal{P}_\varphi(M), \mathrm{ep}(\mathbb{Q}') = s\} + s = I(s) + s, \end{aligned}$$

where we used the fact that $\mathbb{Q} \in \mathcal{P}_\varphi(M)$ if and only if $\widehat{\mathbb{Q}} \in \mathcal{P}_\varphi(M)$. This completes the proof of Theorem 3.9. □

4 Weak Gibbs measures

Definition 4.1 *We say that $\mathbb{P} \in \mathcal{P}(M)$ is a* weak Gibbs measure *for $\mathcal{G} \in \mathcal{A}(M)$ if for any $n \geq 1$ and any $\epsilon > 0$ there is $K_n(\epsilon) \geq 1$ such that*

$$K_n(\epsilon)^{-1}\, \mathrm{e}^{G_n(x) - n\mathfrak{p}_\varphi(\mathcal{G})} \leq \mathbb{P}\big(B_n(x,\epsilon)\big) \leq K_n(\epsilon)\, \mathrm{e}^{G_n(x) - n\mathfrak{p}_\varphi(\mathcal{G})}, \tag{4.1}$$

$$\lim_{\epsilon \downarrow 0} \limsup_{n\to\infty} \frac{1}{n} \log K_n(\epsilon) = 0, \tag{4.2}$$

where (4.1) holds for every $x \in M$.

Remark 4.2 It is easy to see that if a probability measure $\mathbb{P} \in \mathcal{P}(M)$ is weak Gibbs for two potential sequences $\mathcal{G}, \mathcal{G}' \in \mathcal{A}(M)$, then $(G_n - n\mathfrak{p}_\varphi(\mathcal{G}))_{n\geq 1}$ and $(G_n' - n\mathfrak{p}_\varphi(\mathcal{G}'))_{n\geq 1}$ are equivalent in the sense of Remark 0.11. Conversely, if $\mathcal{G} \sim \mathcal{G}'$, then $\mathbb{P}$ is weak Gibbs for $\mathcal{G}$ iff it is weak Gibbs for $\mathcal{G}'$.

We emphasize that the definition of weak Gibbs measure does not require $\mathbb{P} \in \mathcal{P}_\varphi(M)$. Notice also that it follows from (4.1) that the support of $\mathbb{P}$ coincides with M. The following lemma shows that if the latter property is satisfied, then it suffices to require the validity of (4.1) almost everywhere. This observation is technically useful when the transfer operators are used to construct weak Gibbs measures; see [Kes01, Section 2], [Cli10, Appendix B], and [CJPS18].

Lemma 4.3 *Let $\mathcal{G} = \{G_n\} \in \mathcal{A}(M)$, and let $\mathbb{P} \in \mathcal{P}(M)$ be a measure such that* $\mathrm{supp}(\mathbb{P}) = M$. *Assume that for all $\epsilon > 0$, (4.1) holds for $\mathbb{P}$-almost every $x \in M$, with $\{K_n(\epsilon)\}$ satisfying (4.2). Then $\mathbb{P}$ is weak Gibbs for $\mathcal{G}$.*

Proof. Define

$$\gamma(n,\epsilon) := \sup_{x\in M} \sup_{y,z \in B_n(x,\epsilon)} \frac{1}{n}\big|G_n(y) - G_n(z)\big|. \tag{4.3}$$

By (2.4),

$$\lim_{\epsilon\downarrow 0} \limsup_{n\to\infty} \gamma(n,\epsilon) = 0. \tag{4.4}$$

For given $n \geq 1$ and $\epsilon > 0$, let $A \subset M$ be a set of full $\mathbb{P}$-measure on which (4.1) holds and let $x \in M$ be an arbitrary point. Since $\overline{A} = M$, we can find $x' \in A \cap B_n(x, \epsilon/2)$, so that

$$\mathbb{P}(B_n(x,\epsilon)) \geq \mathbb{P}(B_n(x',\epsilon/2)) \geq K_n(\epsilon/2)\, e^{G_n(x') - n\mathfrak{p}_\varphi(\mathcal{G})} \geq K_n'(\epsilon)^{-1} e^{G_n(x) - n\mathfrak{p}_\varphi(\mathcal{G})},$$

$$\mathbb{P}(B_n(x,\epsilon)) \leq \mathbb{P}(B_n(x',2\epsilon)) \leq K_n(2\epsilon) e^{G_n(x') - n\mathfrak{p}_\varphi(\mathcal{G})} \leq K_n'(\epsilon) e^{G_n(x) - n\mathfrak{p}_\varphi(\mathcal{G})},$$

where

$$K_n'(\epsilon) = e^{n\gamma(n,\epsilon)} \max(K_n(2\epsilon), K_n(\epsilon/2)).$$

Relation (4.4) gives that

$$\lim_{\epsilon \downarrow 0} \limsup_{n \to \infty} \frac{1}{n} \log K_n'(\epsilon) = 0$$

and the statement follows. □

The following result is again a special case of Theorem 5.2.

Theorem 4.4 *Assume that Conditions* **(USCE)** *and* **(S)** *hold and that $\mathbb{P}$ is a weak Gibbs measure for $\mathcal{G} \in \mathcal{A}(M)$. Then the conclusions of Theorem 3.1 hold with $\mathbb{P}_n$ replaced with $\mathbb{P}$.*

Recall that σ_n is defined by (3.9). By Lemma 5.4 below, the limit

$$e(\alpha) := \lim_{n \to \infty} \frac{1}{n} \log \int_M \mathrm{e}^{-\alpha \sigma_n} \mathrm{d}\mathbb{P}$$

exists for all $\alpha \in \mathbb{R}$ and is given by

$$e(\alpha) = \mathfrak{p}_\varphi((1-\alpha)\mathcal{G} + \alpha \mathcal{G} \circ \theta).$$

The proof of the following result is exactly the same as that of Theorem 3.9.

Theorem 4.5 *If in addition to the hypotheses of Theorem 4.4, Condition $(\mathcal{C})/(\mathcal{R})$ holds, then all the conclusions of Theorem 3.9 hold with $\mathbb{P}_n$ replaced with $\mathbb{P}$.*

Remark 4.6 Weak Gibbs measures have been extensively studied in the recent literature on multifractal formalism; see [CJPS18] for references and additional information.

5 Large deviation principles

5.1 Main result and applications

In this subsection we establish the LDP for the empirical measures $\{\mu_n^x\}$ defined in (0.4) and for asymptotically additive potential sequences under some assumptions that cover both the sequence measures (0.14) concentrated on periodic trajectories and the weak Gibbs measures.

We begin with the identification of the rate function for the LDP. Given $\mathcal{G} = \{G_n\} \in \mathcal{A}(M)$ and $V \in C(M)$, we set $\mathcal{G}_V = \{G_n + S_n(V)\}$ and define a map $\mathbb{I} : \mathcal{P}(M) \to [0, \infty]$ by

$$\mathbb{I}(\mathbb{Q}) = \sup_{V \in C(M)} \big(\langle V, \mathbb{Q} \rangle - \mathfrak{p}_\varphi(\mathcal{G}_V) + \mathfrak{p}_\varphi(\mathcal{G}) \big). \tag{5.1}$$

Note that it follows from (2.13) that for any fixed $\mathcal{G}$ we have

$$|\mathfrak{p}_\varphi(\mathcal{G}_V) - \mathfrak{p}_\varphi(\mathcal{G}_{V'})| \le \|V - V'\|_\infty,$$

and in particular that the map $V \mapsto \mathfrak{p}_\varphi(\mathcal{G}_V)$ is continuous.

Lemma 5.1 *The map $\mathcal{P}(M) \ni \mathbb{Q} \mapsto \mathbb{I}(\mathbb{Q}) \in [0, +\infty]$ is lower semicontinuous and convex. Moreover, if Condition* **(USCE)** *holds, then*

$$\mathbb{I}(\mathbb{Q}) = \begin{cases} -\mathcal{G}(\mathbb{Q}) - h_\varphi(\mathbb{Q}) + \mathfrak{p}_\varphi(\mathcal{G}) & \text{for } \mathbb{Q} \in \mathcal{P}_\varphi(M), \\ +\infty & \text{otherwise.} \end{cases} \tag{5.2}$$

An immediate consequence of (5.2) is that the map $\mathcal{P}_\varphi(M) \ni \mathbb{Q} \mapsto \mathbb{I}(\mathbb{Q})$ is affine if φ satisfies **(USCE)**.

Proof. By definition (5.1), the function $\mathbb{I}$ is the pointwise supremum of a family of continuous affine maps. Therefore it is convex and lower semicontinuous.

If $\mathbb{Q} \notin \mathcal{P}_\varphi(M)$, then there is $V \in C(M)$ such that $\delta := \langle V, \mathbb{Q}\rangle - \langle V \circ \varphi, \mathbb{Q}\rangle > 0$. Thus, letting $V_m = m(V - V \circ \varphi)$ and observing that $\|S_n(V_m)\|_\infty \le 2m\|V\|_\infty$, we deduce from (2.13) that $\mathfrak{p}_\varphi(\mathcal{G}_{V_m}) = \mathfrak{p}_\varphi(\mathcal{G})$. Since $\langle V_m, \mathbb{Q}\rangle = m\delta$, we conclude that the supremum in (5.1) is equal to $+\infty$.

To prove (5.2) in the case $\mathbb{Q} \in \mathcal{P}_\varphi(M)$, we rewrite (5.1) as

$$\mathbb{I}(\mathbb{Q}) = \sup_{V \in C(M)} \big(\mathcal{G}_V(\mathbb{Q}) - \mathfrak{p}_\varphi(\mathcal{G}_V)\big) - \mathcal{G}(\mathbb{Q}) + \mathfrak{p}_\varphi(\mathcal{G}).$$

The required result now follows by the variational principle (2.16). □

Given a sequence $\{\mathbb{P}_n\}_{n\ge 1} \subset \mathcal{P}(M)$ and a function $V \in C(M)$, we define

$$A_n(V) = \int_M \mathrm{e}^{n\langle V, \mu_n^x\rangle} \mathbb{P}_n(\mathrm{d}x) = \int_M \mathrm{e}^{S_n(V)(x)} \mathbb{P}_n(\mathrm{d}x).$$

Theorem 5.2 *Suppose that Conditions* **(S)** *and* **(USCE)** *hold. Let $\{\mathbb{P}_n\} \subset \mathcal{P}(M)$ and $\mathcal{G} \in \mathcal{A}(M)$. Suppose also that the following holds:*

(C1) *For all $V \in C(M)$, we have*

$$\lim_{n\to\infty} \frac{1}{n} \log A_n(V) = \mathfrak{p}_\varphi(\mathcal{G}_V) - \mathfrak{p}_\varphi(\mathcal{G}). \tag{5.3}$$

(C2) *For any $0 < \delta \ll 1$ there is an integer $n_0(\delta) \ge 1$ and sequences $K_n(\delta) \ge 1$, $m_\delta(n) \in \mathbb{N}$, such that*

$$K_n(\delta)^{-1}\, \mathrm{e}^{G_n(x) - n\mathfrak{p}_\varphi(\mathcal{G})} \le \mathbb{P}_n\big(B_{n-m_\delta(n)}(x, \delta)\big) \quad \text{for } x \in M,\ n \ge n_0(\delta), \tag{5.4}$$

$$\lim_{\delta\downarrow 0} \limsup_{n\to\infty} \frac{1}{n}\big(\log K_n(\delta) + m_\delta(n)\big) = 0. \tag{5.5}$$

Then:

(1) *The LDP (3.2) holds with the rate function $\mathbb{I}$ given by (5.2).*

(2) *For any* $\mathcal{V} = \{V_n\} \in \mathcal{A}(M)$*, the sequence* $\frac{1}{n}V_n$ *satisfies the LDP* (3.4) *with the good convex rate function* $I : \mathbb{R} \to [0, +\infty]$ *defined by the contraction relation*

$$I(s) = \inf\{\mathbb{I}(\mathbb{Q}) : \mathbb{Q} \in \mathcal{P}_\varphi(M), \mathcal{V}(\mathbb{Q}) = s\}. \tag{5.6}$$

Moreover, I *is the Legendre transform of the function* $\alpha \mapsto \mathfrak{p}_\varphi(\mathcal{G} + \alpha\mathcal{V}) - \mathfrak{p}_\varphi(\mathcal{G})$.

The two parts of Theorem 5.2 are proved in Sections 5.2 and 5.3 below. Here we prove that Theorem 5.2 applies to the sequences (0.14) and to weak Gibbs measures.

Lemma 5.3 *Assume that* φ *satisfies Conditions* **(WPS)** *and* **(PAP)**. *Then the measures defined by* (0.14) *satisfy Conditions* (C1) *and* (C2) *of Theorem 5.2.*

Proof. Since

$$\log A_n = \log \sum_{x \in M_n} \mathrm{e}^{G_n(x) + S_n V(x)} - \log Z_n = \log \sum_{x \in M_n} \mathrm{e}^{G_n(x) + S_n V(x)} - \log \sum_{x \in M_n} \mathrm{e}^{G_n(x)},$$

(PAP) yields (5.3) and (C1).

To prove (C2), let $m_\delta(n)$ and $n_0(\delta)$ be as in the definition of **(WPS)**. Set

$$\lambda(n, \delta) = \frac{1}{n}\|G_{n - m_\delta(n)} - G_n\|_\infty, \tag{5.7}$$

and let $\gamma(n, \delta)$ be defined by (4.3). It follows from Lemma 2.4 that

$$\lim_{\delta \downarrow 0} \limsup_{n \to \infty} \big(\gamma(n, \delta) + \lambda(n, \delta)\big) = 0. \tag{5.8}$$

By **(WPS)**, for any $n \geq n_0(\delta)$ the intersection $M_n \cap B_{n - m_\delta(n)}(x, \delta)$ contains at least one point x_n, so that, by writing $n'(\delta) = n - m_\delta(n)$, we have

$$\begin{aligned}
\log \mathbb{P}_n\big(B_{n'(\delta)}(x, \delta)\big) &\geq G_{n'(\delta)}(x_n) - \log Z_{n'(\delta)} \\
&\geq G_{n'(\delta)}(x) - n'(\delta)\gamma(n'(\delta), \delta) - \log Z_{n'(\delta)} \\
&\geq G_n(x) - n\lambda(n, \delta) - n'(\delta)\gamma(n'(\delta), \delta) - \log Z_{n'(\delta)} \\
&\geq G_n(x) - n\mathfrak{p}_\varphi(\mathcal{G}) - \log K_n(\delta),
\end{aligned}$$

where

$$K_n(\delta) = \exp\big\{n\lambda(n, \delta) + n'(\delta)\gamma(n'(\delta), \delta) + |\log Z_{n'(\delta)} - n\mathfrak{p}_\varphi(\mathcal{G})|\big\}.$$

Condition (C2) now follows from (5.8) and the definition of topological pressure. □

Lemma 5.4 *Suppose that Condition* **(USCE)** *holds and that* $\mathbb{P} \in \mathcal{P}(M)$ *is a weak Gibbs measure for* $\mathcal{G} \in \mathcal{A}(M)$. *Then for all* $\mathcal{G}' \in \mathcal{A}(M)$ *we have*

$$\lim_{n \to \infty} \frac{1}{n} \log \left\langle \mathrm{e}^{G'_n}, \mathbb{P} \right\rangle = \mathfrak{p}_\varphi(\mathcal{G} + \mathcal{G}') - \mathfrak{p}_\varphi(\mathcal{G}). \tag{5.9}$$

Proof. The proof follows Proposition 3.2 in [Kif90] (see also Proposition 10.3 in [JPRB11]); since an additional limiting argument is needed, we include it for the sake of completeness.

For any $\epsilon > 0$, $n \geq 1$, and any (ϵ, n)-spanning set $E_{\epsilon,n}$, using (4.1) we derive

$$\begin{aligned}\left\langle \mathrm{e}^{G'_n}, \mathbb{P}\right\rangle &\leq \sum_{x\in E_{\epsilon,n}} \langle \mathbf{1}_{B_n(x,\epsilon)}\mathrm{e}^{G'_n}, \mathbb{P}\rangle \leq \mathrm{e}^{n\gamma(n,\epsilon)} \sum_{x\in E_{\epsilon,n}} \mathrm{e}^{G'_n(x)}\mathbb{P}\big(B_n(x,\epsilon)\big) \\ &\leq K_n(\epsilon)\mathrm{e}^{n\gamma(n,\epsilon)-n\mathfrak{p}_\varphi(\mathcal{G})} \sum_{x\in E_{\epsilon,n}} \mathrm{e}^{G_n(x)+G'_n(x)},\end{aligned}$$

where $\gamma(n, \epsilon)$ is defined by (4.3). It follows that

$$\limsup_{n\to\infty} \frac{1}{n} \log \left\langle \mathrm{e}^{G'_n}, \mathbb{P}\right\rangle \leq \limsup_{n\to\infty}\big(n^{-1}\log K_n(\epsilon) + \gamma(n,\epsilon) + n^{-1}\log S(\mathcal{G}+\mathcal{G}', \epsilon, n)\big) - \mathfrak{p}_\varphi(\mathcal{G}).$$

Sending $\epsilon \to 0$ and using expression (2.9) for $\mathfrak{p}_\varphi(\mathcal{G} + \mathcal{G}')$, we obtain the "$\leq$" inequality in (5.9).

To prove the other inequality, we proceed similarly, observing that for any (ϵ, n)-separated set $E_{\epsilon,n}$ we have

$$\left\langle \mathrm{e}^{G'_n}, \mathbb{P}\right\rangle \geq K_n^{-1}(\epsilon)e^{-n\gamma(n,\epsilon)-n\mathfrak{p}_\varphi(\mathcal{G})} \sum_{x\in E_{\epsilon,n}} \mathrm{e}^{G_n(x)+G'_n(x)}.$$

Taking the supremum over all (ϵ, n)-separated sets, repeating the above argument, and using expression (2.10) for $\mathfrak{p}_\varphi(\mathcal{G} + \mathcal{G}')$, we obtain the desired result. □

Lemma 5.5 *Let $\mathbb{P} \in \mathcal{P}(M)$ be a weak Gibbs measure. Then Conditions* (C1) *and* (C2) *of Theorem 5.2 hold for $\mathbb{P}_n = \mathbb{P}$.*

Proof. (C1) follows from the special case $\mathcal{G}' = \{S_nV\}$ in Lemma 5.4. (C2) with $m_\delta(n) \equiv 0$ follows from Definition 4.1. □

5.2 Proof of the LDP for empirical measures

We first give the proof of Theorem 5.2 (1), which is completed in the following two steps.

Step 1: LD upper bound.

Proposition 5.6 *If Condition* **(USCE)** *holds, then*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}_n\{\mu_n^{\cdot} \in F\} \leq -\inf_{\mathbb{Q}\in F} \mathbb{I}(\mathbb{Q}) \tag{5.10}$$

holds for any closed set $F \subset \mathcal{P}(M)$.

Proof. It is a well-known fact (see [Kif90, Theorem 2.1] and also [DZ00, Section 4.5.1]) that the existence of limit (5.3) implies inequality (5.10) with a rate function $\mathbb{I}$ given by the Legendre transform of $\mathfrak{p}_\varphi(\mathcal{G}_V) - \mathfrak{p}_\varphi(\mathcal{G})$ with respect to V, *i.e.*, by (5.1). □

Step 2: LD lower bound. We need to prove that, for any open set $O \subset \mathcal{P}(M)$,

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}_n\{\mu_n^{\cdot} \in O\} \geq -\inf_{\mathbb{Q}\in O} \mathbb{I}(\mathbb{Q}).$$

This inequality will be established if we prove that, for any $\mathbb{Q} \in O$,

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}_n\{\mu_n^{\cdot} \in O\} \geq -\mathbb{I}(\mathbb{Q}). \tag{5.11}$$

Moreover, we only need to consider $\mathbb{Q} \in \mathcal{P}_\varphi(M)$ since otherwise $\mathbb{I}(\mathbb{Q}) = +\infty$, and (5.11) is trivially satisfied. To prove (5.11) for $\mathbb{Q} \in \mathcal{P}_\varphi(M)$, we follow a strategy that goes back to [FO88], see also [EKW94, PS05], and first consider the special case $\mathbb{Q} \in \mathcal{E}_\varphi(M)$. In the argument we shall need the following consequence of Birkhoff's ergodic theorem (cf. [FH10, Proposition A.1]).

Lemma 5.7 *Let $\mathbb{Q} \in \mathcal{E}_\varphi(M)$ and $\mathcal{G} \in \mathcal{A}(M)$. Then, for $\mathbb{Q}$-almost every $x \in M$ we have*

$$\lim_{n\to\infty} \frac{1}{n} G_n(x) = \mathcal{G}(\mathbb{Q}).$$

Proof. Let $\{G^{(k)}\}$ be an approximating sequence for $\mathcal{G}$. By Birkhoff's ergodic theorem, for any $k \geq 1$ we have

$$\lim_{n\to\infty} \frac{1}{n} S_n G^{(k)}(x) = \langle G^{(k)}, \mathbb{Q}\rangle$$

for $\mathbb{Q}$-almost every $x \in M$. By (2.2), the right-hand side converges to $\mathcal{G}(\mathbb{Q})$ when $k \to \infty$, and (0.11) completes the proof. □

Proposition 5.8 *Assume (5.4). Then inequality (5.11) holds for any open set $O \subset \mathcal{P}(M)$ and any $\mathbb{Q} \in O \cap \mathcal{E}_\varphi(M)$.*

Proof. Fix $O \subset \mathcal{P}(M)$ and $\mathbb{Q} \in O \cap \mathcal{E}_\varphi(M)$. Given $V_1, \ldots, V_m \in C(M)$ and $\epsilon > 0$ we set

$$R_\epsilon = R_\epsilon(V_1, \ldots, V_m) = \bigcap_{j=1}^{m} \big\{\mathbb{Q}' \in \mathcal{P}(M) : |\langle V_j, \mathbb{Q}'\rangle - \langle V_j, \mathbb{Q}\rangle| < \epsilon\big\}.$$

Since $O \ni \mathbb{Q}$ is open, we can find finitely many functions $V_1, \ldots, V_m \in C(M)$ and a number $\epsilon_0 > 0$ such that $R_{2\epsilon_0} \subset O$. Let $\epsilon \in]0, \epsilon_0[$ and let X_n^ϵ be the set of points $x \in M$ such that

$$|n^{-1}G_n(x) - \mathcal{G}(\mathbb{Q})| < \epsilon, \quad |n^{-1}S_n V_j(x) - \langle V_j, \mathbb{Q}\rangle| < \epsilon \quad \text{for } j = 1, \ldots, m.$$

Note that

$$X_n^\epsilon \subset X_n^{2\epsilon} \subset \{x \in M : \mu_n^x \in R_{2\epsilon}\} \subset \{x \in M : \mu_n^x \in O\}.$$

Since $\mathbb{Q}$ is ergodic, it follows from Lemma 5.7 that

$$\lim_{n\to\infty} \mathbb{Q}(X_n^\epsilon) = 1. \tag{5.12}$$

For $\delta > 0$ and $n \geq n_0(\delta)$ we define

$$Y_n^\epsilon(\delta) = \big\{x \in M : \mathbb{Q}(B_{n-m_{\delta/2}(n)}(x, \delta)) \leq e^{-n(h_\varphi(\mathbb{Q})-\epsilon)}\big\}.$$

Invoking the ergodicity of $\mathbb{Q}$, the Brin–Katok local entropy formula (see [BK83]) implies that

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{Q}\big(B_{n-m_{\delta/2}(n)}(x,\delta)\big) = -h_\varphi(\mathbb{Q}) \quad \text{for } \mathbb{Q}\text{-almost every } x\in M.$$

(We have used also that $\lim_{\delta\downarrow 0}\liminf_{n\to\infty}\frac{n-m_{\delta/2}(n)}{n} = \lim_{\delta\downarrow 0}\limsup_{n\to\infty}\frac{n-m_{\delta/2}(n)}{n} = 1$.) Combining this with a simple measure-theoretic argument (similar to the one used to prove Egorov's theorem), we see that

$$\lim_{\delta\downarrow 0}\liminf_{n\to\infty}\mathbb{Q}\big(Y_n^\epsilon(\delta)\big) = 1. \tag{5.13}$$

It follows from (5.12) and (5.13) that for all small enough $\delta > 0$ there is an integer $n_1(\delta)\geq n_0(\delta)$ such that

$$\mathbb{Q}\big(X_n^\epsilon\cap Y_n^\epsilon(\delta)\big)\geq \tfrac{1}{2} \quad \text{for all } n\geq n_1. \tag{5.14}$$

Moreover, by (2.6) (applied to G_n and to the potential sequences $\{S_nV_j\}$) and (5.5), we can assume, by possibly decreasing δ and increasing n_1, that for all $n\geq n_1$,

$$\sup_{x\in M}\sup_{y,z\in B_{n-m_{\delta/2}(n)}(x,\delta/2)}\frac{1}{n}\big|G_n(y)-G_n(z)\big| < \epsilon, \tag{5.15}$$

$$\sup_{x\in M}\sup_{y,z\in B_{n-m_{\delta/2}(n)}(x,\delta/2)}\frac{1}{n}\big|S_nV_j(y)-S_nV_j(z)\big| < \epsilon, \quad \text{for } j=1,\dots,m, \tag{5.16}$$

$$\frac{1}{n}\log K_n(\delta/2) < \epsilon. \tag{5.17}$$

Suppose that, for sufficiently large n, we have constructed points $x_1,\dots,x_{r_n}\in X_n^\epsilon\cap Y_n^\epsilon(\delta)$ such that the balls $B_{n-m_{\delta/2}(n)}(x_i,\delta/2)$ are pairwise disjoint, and

$$\bigcup_{i=1}^{r_n} B_{n-m_{\delta/2}(n)}(x_i,\delta/2)\subset X_n^{2\epsilon}, \qquad X_n^\epsilon\cap Y_n^\epsilon(\delta)\subset\bigcup_{i=1}^{r_n} B_{n-m_{\delta/2}(n)}(x_i,\delta). \tag{5.18}$$

In this case, we can write

$$\begin{aligned}\mathbb{P}_n\{\mu_n^{\cdot}\in O\}\geq\mathbb{P}_n(X_n^{2\epsilon})&\geq\sum_{i=1}^{r_n}\mathbb{P}_n\big(B_{n-m_{\delta/2}(n)}(x_i,\delta/2)\big)\\ &=\sum_{i=1}^{r_n}\frac{\mathbb{P}_n\big(B_{n-m_{\delta/2}(n)}(x_i,\delta/2)\big)}{\mathbb{Q}\big(B_{n-m_{\delta/2}(n)}(x_i,\delta)\big)}\,\mathbb{Q}\big(B_{n-m_{\delta/2}(n)}(x_i,\delta)\big).\end{aligned} \tag{5.19}$$

Since $x_i\in Y_n^\epsilon(\delta)$, we have

$$\mathbb{Q}\big(B_{n-m_{\delta/2}(n)}(x_i,\delta)\big)\leq \mathrm{e}^{-n(h_\varphi(\mathbb{Q})-\epsilon)} \tag{5.20}$$

for $n\geq n_0$. Moreover, (5.4) and $x_i\in X_n^\epsilon$ imply that

$$\begin{aligned}\mathbb{P}_n\big(B_{n-m_{\delta/2}(n)}(x_i,\delta/2)\big)&\geq K_n(\delta/2)^{-1}\mathrm{e}^{G_n(x_i)-n\mathfrak{p}_\varphi(\mathcal{G})}\\ &\geq K_n(\delta/2)^{-1}\mathrm{e}^{n\mathcal{G}(\mathbb{Q})-n\epsilon-n\mathfrak{p}_\varphi(\mathcal{G})}.\end{aligned} \tag{5.21}$$

Thus, using (5.19), (5.20), (5.21), the second inclusion in (5.18), and (5.14), we derive

$$\begin{aligned}\mathbb{P}_n\{\mu_n^{\boldsymbol{\cdot}} \in O\} &\geq \frac{1}{2}K_n(\delta/2)^{-1}\mathrm{e}^{n(\mathcal{G}(\mathbb{Q})+h_\varphi(\mathbb{Q}))-2n\epsilon-n\mathfrak{p}_\varphi(\mathcal{G})} \\ &\geq \frac{1}{2}K_n(\delta/2)^{-1}\mathrm{e}^{-n\mathbb{I}(\mathbb{Q})-2n\epsilon},\end{aligned}$$

where the second inequality follows from (5.2). Together with (5.17) this gives

$$\liminf_{n\to\infty} n^{-1}\log\mathbb{P}_n\{\mu_n^{\boldsymbol{\cdot}} \in O\} \geq -\mathbb{I}(\mathbb{Q}) - 2\epsilon.$$

Since $\epsilon \in]0,\epsilon_0[$ can be chosen arbitrarily small, (5.11) follows.

It remains to construct points $x_1,\dots,x_{r_n} \in X_n^\epsilon \cap Y_n^\epsilon(\delta)$ such that the balls $B_{n-m_{\delta/2}(n)}(x_i,\delta/2)$ are disjoint and (5.18) holds.

First, it follows from (5.15) and (5.16) that for all $y \in B_{n-m_{\delta/2}(n)}(x,\delta/2)$,

$$\begin{aligned}\left|n^{-1}G_n(y)-\mathcal{G}(\mathbb{Q})\right| &\leq \epsilon + \left|n^{-1}G_n(x)-\mathcal{G}(\mathbb{Q})\right| < 2\epsilon, \\ \left|n^{-1}S_nV_j(y)-\langle V_j,\mathbb{Q}\rangle\right| &\leq \epsilon + \left|n^{-1}S_nV_j(x)-\langle V_j,\mathbb{Q}\rangle\right| < 2\epsilon \quad \text{for } j=1,\dots,m,\end{aligned}$$

and so

$$x \in X_n^\epsilon \Longrightarrow B_{n-m_{\delta/2}(n)}(x,\delta/2) \subset X_n^{2\epsilon} \tag{5.22}$$

for all $n \geq n_1$. Now let $\mathfrak{B} = \{B_{n-m_{\delta/2}(n)}(x_i,\delta/2) : 1 \leq i \leq r_n\}$ be any maximal[20] collection of disjoint balls included in $X_n^{2\epsilon}$ such that $x_i \in X_n^\epsilon \cap Y_n^\epsilon(\delta)$. The first inclusion in (5.18) follows from (5.22). To prove the second one, suppose that $x_* \in X_n^\epsilon \cap Y_n^\epsilon(\delta)$ does not belong to any of the balls $B_{n-m_{\delta/2}(n)}(x_i,\delta)$. Then $B_{n-m_{\delta/2}(n)}(x_*,\delta/2)$ does not intersect the balls in $\mathfrak{B}$ and, by (5.22), is included in $X_n^{2\epsilon}$, and the collection $\mathfrak{B}$ is not maximal. This completes the proof of the proposition. □

The following proposition completes the proof of Part (1) of Theorem 5.2.

Proposition 5.9 *If, in addition to the hypotheses of Proposition 5.8, Condition* **(S)** *holds, then inequality* (5.11) *holds for any open set $O \subset \mathcal{P}(M)$ and any $\mathbb{Q} \in O \cap \mathcal{P}_\varphi(M)$.*

Proof. Let $\mathbb{Q} \in O \cap \mathcal{P}_\varphi(M)$. By Proposition 2.2, there exists a sequence $\{\mathbb{Q}^{(m)}\} \subset \mathcal{E}_\varphi(M)$ such that

$$\mathbb{Q}^{(m)} \rightharpoonup \mathbb{Q}, \qquad h_\varphi(\mathbb{Q}^{(m)}) \to h_\varphi(\mathbb{Q}),$$

as $m \to \infty$. In this case, in view of (5.2) and the continuity assertion in Lemma 2.3, we have

$$\mathbb{I}(\mathbb{Q}^{(m)}) = -\mathcal{G}(\mathbb{Q}^{(m)}) - h_\varphi(\mathbb{Q}^{(m)}) + \mathfrak{p}_\varphi(\mathcal{G}) \to -\mathcal{G}(\mathbb{Q}) - h_\varphi(\mathbb{Q}) + \mathfrak{p}_\varphi(\mathcal{G}) = \mathbb{I}(\mathbb{Q}) \tag{5.23}$$

as $m \to \infty$. Since O is open, $\mathbb{Q}^{(m)} \in O$ for large enough m, and by Proposition 5.8, we have

$$\liminf_{n\to\infty} n^{-1}\log\mathbb{P}_n\{\mu_n^{\boldsymbol{\cdot}} \in O\} \geq -\mathbb{I}(\mathbb{Q}^{(m)}).$$

Passing to the limit $m \to \infty$ and using (5.23) we obtain inequality (5.11). □

[20] Such a maximal collection exists, since (5.4) gives an absolute upper bound on r_n.

5.3 Proof of the LDP for asymptotically additive sequences of functions

Part (2) of Theorem 5.2 is a special case of Theorem 4.2.23 in [DZ00].[21] For the reader's convenience, we outline the proof in our case.

Let $\mathcal{V} = \{V_n\} \in \mathcal{A}(M)$ with approximating sequence $\{V^{(k)}\}$. We define the random variables $\xi_n = \frac{1}{n}V_n$ and

$$\xi_n^k := \frac{1}{n}S_nV^{(k)} = \langle V^{(k)}, \mu_n^{\cdot}\rangle,$$

and consider them under the law $\mathbb{P}_n$. By the definition of $V^{(k)}$ we have

$$\lim_{k\to\infty}\limsup_{n\to\infty}\|\xi_n^k - \xi_n\|_\infty = 0. \tag{5.24}$$

By the usual contraction principle [DZ00, Theorem 4.2.1], for each k the family $\{\xi_n^k\}_{n\geq 1}$ satisfies the LDP with the good convex rate function

$$I_k(s) = \inf\{\mathbb{I}(\mathbb{Q}) : \mathbb{Q} \in \mathcal{P}_\varphi(M), \langle V^{(k)}, \mathbb{Q}\rangle = s\}. \tag{5.25}$$

We now show that ξ_n satisfies the LDP with the rate function

$$I(s) := \sup_{\delta>0}\liminf_{k\to\infty}\inf_{z\in B(s,\delta)} I_k(z).$$

It is immediate that I is lower semicontinuous. By (5.24), for all $\delta > 0$,

$$\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}_n\{\xi_n \in B(s,\delta)\} \leq -\liminf_{k\to\infty}\inf_{y\in B(s,2\delta)} I_k(y),$$
$$\liminf_{n\to\infty}\frac{1}{n}\log\mathbb{P}_n\{\xi_n \in B(s,\delta)\} \geq -\liminf_{k\to\infty}\inf_{y\in B(s,\delta/2)} I_k(y).$$

Hence we obtain

$$\sup_{\delta>0}\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}_n\{\xi_n \in B(s,\delta)\} = \sup_{\delta>0}\liminf_{n\to\infty}\frac{1}{n}\log\mathbb{P}_n\{\xi_n \in B(s,\delta)\} = I(s).$$

A standard argument [DZ00, Theorem 4.1.11] implies that ξ_n satisfies the weak LDP with rate function I. Since the family $\{\xi_n\}$ is bounded (recall Remark 0.11), ξ_n actually satisfies the full LDP, and I is a good rate function. It remains to show that $I(s) = J(s)$, where

$$J(s) = \inf\{\mathbb{I}(\mathbb{Q}) : \mathbb{Q} \in \mathcal{P}_\varphi(M), \mathcal{V}(\mathbb{Q}) = s\}.$$

Note that J is lower semicontinuous, since $\mathbb{Q} \mapsto \mathcal{V}(\mathbb{Q})$ is continuous on $\mathcal{P}_\varphi(M)$ by Lemma 2.3, and is obviously convex. It follows from (5.25) that

$$I(s) = \sup_{\delta>0}\liminf_{k\to\infty}\inf\{\mathbb{I}(\mathbb{Q}) : \mathbb{Q} \in \mathcal{P}_\varphi(M), \langle V^{(k)}, \mathbb{Q}\rangle \in B(s,\delta)\},$$

while the lower semicontinuity of J gives

$$J(s) = \sup_{\delta>0}\inf_{y\in B(s,\delta)} J(s) = \sup_{\delta>0}\inf\{\mathbb{I}(\mathbb{Q}) : \mathbb{Q} \in \mathcal{P}_\varphi(M), \mathcal{V}(\mathbb{Q}) \in B(s,\delta)\}.$$

Using that $\langle V^{(k)}, \mathbb{Q}\rangle \to \mathcal{V}(\mathbb{Q})$ uniformly on $\mathcal{P}_\varphi(M)$ (recall (2.2)), we derive that $J(s) = I(s)$. This completes the proof of Part (2) of Theorem 5.2.

[21] In the notation therein, $\mathcal{X} = \mathcal{P}(M)$, $\mathcal{Y} = \mathbb{R}$, $f_m(\mathbb{Q}) = \langle V^{(m)}, \mathbb{Q}\rangle$ and $f(\mathbb{Q})$ is defined by $\mathcal{V}(\mathbb{Q})$ when $\mathbb{Q} \in \mathcal{P}_\varphi(M)$ and arbitrarily when $\mathbb{Q} \in \mathcal{P}(M) \setminus \mathcal{P}_\varphi(M)$.

6 Conditions for asymptotic additivity

In this section we give some necessary and sufficient conditions for a potential sequence to be asymptotically additive.

A sequence $\mathcal{G} = \{G_n\} \subset B(M)$ is said to have *tempered variation* if

$$\lim_{\epsilon\downarrow 0} \limsup_{n\to\infty} \sup_{x\in M} \sup_{y,z\in B_n(x,\epsilon)} \frac{1}{n}\big|G_n(y) - G_n(z)\big| = 0. \tag{6.1}$$

We have shown in Lemma 2.4 that asymptotic additivity implies (6.1). Below, we shall sometimes take (6.1) as an assumption (along with others), in order to obtain asymptotic additivity.

We recall that $\{G_n\} \subset B(M)$ is called *weakly almost additive* if for all $n, m \geq 1$,

$$-C_m + G_m + G_n \circ \varphi^m \leq G_{m+n} \leq C_m + G_m + G_n \circ \varphi^m, \tag{6.2}$$

where $\lim_{n\to\infty} n^{-1}C_n = 0$.

The main result of this section is

Theorem 6.1 *If $\mathcal{G} = \{G_n\}_{n\geq 1} \subset B(M)$ satisfies any of the following conditions, then $\mathcal{G}$ is asymptotically additive.*

(1) $G_n = S_n G$ for each n, with $G \in C(M)$.

(2) $G_n = S_n G$ for each n, with $G \in B(M)$, and $\mathcal{G}$ has tempered variation.

(3) $\mathcal{G}$ is weakly almost additive, and $G_n \in C(M)$ for each n.

(4) $\mathcal{G}$ is weakly almost additive and has tempered variation.

Moreover, if $\mathcal{G} \subset C(M)$, then the following assertion is equivalent to asymptotic additivity of $\mathcal{G}$.

(5) $\mathcal{G}$ satisfies

$$\lim_{k\to\infty} \limsup_{n\to\infty} n^{-1}\Big\|G_n - \frac{1}{k}S_n G_k\Big\|_\infty = 0. \tag{6.3}$$

Finally, for $\mathcal{G} \subset B(M)$, each of the following assertions is equivalent to asymptotic additivity of $\mathcal{G}$.

(6) $\mathcal{G}$ has tempered variation and satisfies (6.3).

(7) $\mathcal{G}$ has tempered variation and there exists a sequence $\{G^{(k)}\} \subset B(M)$ such that

$$\lim_{k\to\infty} \limsup_{n\to\infty} n^{-1}\big\|G_n - S_n G^{(k)}\big\|_\infty = 0. \tag{6.4}$$

(8) There exists a sequence $\{G^{(k)}\} \subset B(M)$ such that (6.4) holds, and

$$\lim_{k\to\infty} \lim_{\epsilon\downarrow 0} \limsup_{n\to\infty} \sup_{x\in M} \sup_{y,z\in B_n(x,\epsilon)} \frac{1}{n}\big|S_n G^{(k)}(y) - S_n G^{(k)}(z)\big| = 0. \tag{6.5}$$

Let us mention that various partial results contained in Theorem 6.1 were known earlier (e.g., see the papers [Bar06, ZZC11, Bar11] and the references therein). However, the equivalence relationships stated above seem to be new.

We start with the following lemma, which was established in [FH10, Proposition A.5] and [ZZC11, Proposition 2.1] in the almost additive case, that is, when $\{C_m\}$ in (6.2) is a constant sequence.

Lemma 6.2 *Assume that $\{G_n\} \subset B(M)$ is weakly almost additive. Then (6.3) holds.*

Proof. Given two positive integers n and k, we write n_k for the integer part of n/k and, for a function V, denote

$$S_n^k V = \sum_{r=0}^{n-1} V \circ \varphi^{rk}.$$

Suppose that for any $\epsilon > 0$ we can find $k_\epsilon \geq 1$ such that

$$\limsup_{n\to\infty} \left\| n^{-1}G_n - n^{-1}S_{n_k}^k G_k \right\|_\infty \leq \epsilon \quad \text{for } k \geq k_\epsilon. \tag{6.6}$$

For a fixed $k \geq k_\epsilon$ and any $\ell \in [\![1, k-1]\!]$, replacing x by $\varphi^\ell(x)$ in (6.6) and using an elementary estimate for the ergodic average, we derive

$$\limsup_{n\to\infty} \left\| n^{-1}G_n \circ \varphi^\ell - n^{-1}S_{(n+\ell)_k}^k G_k \circ \varphi^\ell \right\|_\infty \leq \epsilon \quad \text{for } k \geq k_\epsilon.$$

Combining this with the relation

$$\lim_{n\to\infty} \left\| (n+\ell)^{-1}G_{n+\ell} - n^{-1}G_n \circ \varphi^\ell \right\|_\infty = 0$$

which follows from (6.2), we obtain

$$\limsup_{n\to\infty} \left\| n^{-1}G_n - n^{-1}S_{n_k}^k G_k \circ \varphi^\ell \right\|_\infty \leq \epsilon \quad \text{for } k \geq k_\epsilon. \tag{6.7}$$

Now note that

$$k^{-1}\sum_{\ell=0}^{k-1} S_{n_k}^k G_k \circ \varphi^\ell = k^{-1}S_{kn_k}G_k.$$

Comparing with (6.7) we get that for $k \geq k_\epsilon$,

$$\limsup_{n\to\infty} \left\| n^{-1}G_n - n^{-1}k^{-1}S_{kn_k}G_k \right\|_\infty = \limsup_{n\to\infty} \left\| n^{-1}G_n - n^{-1}k^{-1}S_n G_k \right\|_\infty \leq \epsilon.$$

Since $\epsilon > 0$ is arbitrary, the relation (6.3) follows.

We now prove (6.6). Let us fix an integer $k \geq 1$ and write, for n large, $n = kn_k + \ell$, where $0 \leq \ell \leq k-1$. Applying inequality (6.2) consecutively n_k times, we derive

$$G_n \leq k_n C_k + \sum_{r=0}^{n_k-1} G_k \circ \varphi^{rk} + G_\ell \circ \varphi^{kn_k}.$$

This gives

$$\limsup_{n\to\infty} \sup_{x\in M} \left(n^{-1}G_n(x) - n^{-1}S_{n_k}^k G_k(x) \right) \leq k^{-1}C_k.$$

Replacing G_n by $-G_n$, we derive

$$\liminf_{n\to\infty} \inf_{x\in M} \big(n^{-1}G_n(x) - n^{-1}S_{n_k}^k(G_k)(x)\big) \geq -k^{-1}C_k.$$

Combining the last two inequalities and recalling that $n^{-1}C_n \to 0$, we arrive at (6.6). □

Lemma 6.3 *Let $\mathcal{G} = \{G_n\} \subset B(M)$ be such that there exists $\{G^{(k)}\} \subset B(M)$ satisfying* (6.4)*. Then* (6.3) *holds.*

Proof. For all j we have

$$\begin{aligned}\|G_n - k^{-1}S_nG_k\|_\infty \leq \|G_n - S_nG^{(j)}\|_\infty &+ \|S_nG^{(j)} - k^{-1}S_nS_kG^{(j)}\|_\infty \\ &+ k^{-1}\|S_nS_kG^{(j)} - S_nG_k\|_\infty.\end{aligned}$$

Applying Lemma 6.2 to $\{S_nG^{(j)}\}$ we obtain that for each j,

$$\lim_{k\to\infty}\limsup_{n\to\infty}\frac{1}{n}\|S_nG^{(j)} - k^{-1}S_nS_kG^{(j)}\|_\infty = 0.$$

Fix now $\epsilon > 0$. If j is large enough, then by (6.4) we have

$$\limsup_{n\to\infty}\frac{1}{n}\|G_n - S_nG^{(j)}\|_\infty \leq \epsilon,$$

$$\limsup_{k\to\infty}\limsup_{n\to\infty}\frac{1}{kn}\|S_nS_kG^{(j)} - S_nG_k\|_\infty \leq \limsup_{k\to\infty}\frac{1}{k}\|S_kG^{(j)} - G_k\|_\infty \leq \epsilon.$$

We thus obtain

$$\limsup_{k\to\infty}\limsup_{n\to\infty}\frac{1}{n}\|G_n - k^{-1}S_nG_k\|_\infty \leq 2\epsilon.$$

Since ϵ is arbitrary, this completes the proof. □

Lemma 6.4 *Let $f \in B(M)$ be such that for some fixed n, ϵ, α we have*

$$\sup_{x\in M}\sup_{y\in B_n(x,\epsilon)}|f(y) - f(x)| \leq \alpha.$$

Then there exists a continuous function g such that $\|f - g\|_\infty \leq \alpha$.

Proof. Let $E = \{x_1, x_2, \dots, x_r\}$ be any finite (n, ϵ)-spanning set. Let $\rho_1, \dots, \rho_r$ be a partition of unity subordinated to the collection $\{B_n(x_i, \epsilon) : i = 1, \dots, r\}$ (*i.e.*, ρ_i is continuous, vanishes outside $B_n(x_i, \epsilon)$, and $\sum_i \rho_i = 1$). We claim that the continuous function

$$g = \sum_{i=1}^r \rho_i f(x_i)$$

satisfies the required properties. Let $x \in M$ and let $J \subset \{1, \dots, r\}$ be the largest set such that $x \in \bigcap_{j\in J} B_n(x_j, \epsilon)$. We then have

$$|g(x) - f(x)| = |\sum_{j\in J}\rho_j(x)f(x_j) - f(x)| = |\sum_{j\in J}\rho_j(x)(f(x_j) - f(x))| \leq \alpha,$$

and the result follows. □

Proof of Proposition 6.1. In case (1) we can obviously choose $G^{(k)} = G$ as an approximating sequence for $\mathcal{G}$. Next, (2) is a special case of (4) with $C_n \equiv 0$. In case (3), we obtain by Lemma 6.2 that (6.3) holds so that we are in case (5). In case (4), (6.3) also holds by Lemma 6.2, and hence we find ourselves in the case (6).

That (5) implies asymptotic additivity is immediate, with $G^{(k)} := k^{-1}G_k$ as an approximating sequence. The reverse implication follows immediately from Lemma 6.3 applied to G_n and any approximating sequence $\{G^{(k)}\}$.

We now prove that (6) implies asymptotic additivity. First, it follows from (6.1) that there exists a sequence $\{\epsilon_k\}_{k\geq 1}$ such that

$$\lim_{k\to\infty} \sup_{x\in M} \sup_{y\in B_k(x,\epsilon_k)} \frac{1}{k}|G_k(x) - G_k(y)| = 0. \tag{6.8}$$

Indeed, by (6.1), for each $\ell \in \mathbb{N}$, there exists $\overline{\epsilon}(\ell)$ and $k_0(\ell)$ such that for all $k \geq k_0(\ell)$ we have $\sup_{x\in M} \sup_{y\in B_k(x,\overline{\epsilon}(\ell))} \frac{1}{k}|G_k(x) - G_k(y)| \leq \ell^{-1}$. Let $\{\ell_k\}$ be such that $\ell_k \to \infty$ and $k \geq k_0(\ell_k)$ for all k. Setting $\epsilon_k = \overline{\epsilon}(\ell_k)$ establishes (6.8).

Let $G^{(k)}$ be the regularization of $\frac{1}{k}G_k$ obtained in Lemma 6.4 with respect to the Bowen balls $B_k(x, \epsilon_k)$, with ϵ_k as in (6.8). The function $G^{(k)}$ is continuous and

$$\lim_{k\to\infty} \|k^{-1}G_k - G^{(k)}\|_\infty = 0. \tag{6.9}$$

Since

$$\frac{1}{n}\|G_n - S_nG^{(k)}\|_\infty \leq \frac{1}{n}\|G_n - k^{-1}S_nG_k\|_\infty + \frac{1}{n}\left\|S_n\left[k^{-1}G_k - G^{(k)}\right]\right\|_\infty,$$

relations (6.3) and (6.9) give that G_n is asymptotically additive.

Next, Lemma 6.3 immediately implies that (7) is a special case of (6). Finally, assuming (6.4), it is easy to see that (6.1) and (6.5) are equivalent. Thus, (7) and (8) are equivalent.

We have shown that (8) $\iff$ (7) $\implies$ (6) $\implies \mathcal{G} \in \mathcal{A}(M)$. Since by Lemma 2.4 asymptotic additivity implies (7), the statements (6), (7), (8) are all equivalent to $\mathcal{G} \in \mathcal{A}(M)$. □

Remark 6.5 Note that by the characterization given in (5), if $G_n \in C(M)$ for all n, then the approximating sequence can be chosen to be $G^{(k)} = k^{-1}G_k$. Moreover, the proof gives that when these functions are not continuous, $G^{(k)}$ can be chosen as a regularization of $k^{-1}G_k$. By the finiteness of (0.13), this specific choice of $G^{(k)}$ satisfies $\sup_{k\geq 1} \|G^{(k)}\|_\infty < \infty$ (this is not true of all approximating sequences). Finally, if $\mathcal{G} = \{G_n\} \subset B(M)$ is asymptotically additive, then there exists an asymptotically additive potential sequence $\{G'_n\} \subset C(M)$ in the same class as $\mathcal{G}$ in the sense of Remark 0.11, *i.e.*, such that $\limsup_{n\to\infty} n^{-1}\|G_n - G'_n\|_\infty = 0$. Indeed, it suffices to take an approximating sequence $\{G^{(k)}\} \subset C(M)$ for $\mathcal{G}$, and then to define $G'_n = S_nG^{(k_n)}$ for some well-chosen sequence $k_n \to \infty$ (which is obtained with an argument similar to that leading to (6.8)).

Remark 6.6 The reader may check that $G_n = \log n$ gives a sequence which is weakly almost additive but not almost additive.[22] Moreover, choosing $G_n = \sqrt{n}$ when n is even, and $G_n = 0$ when n is odd,

[22] We recall that a sequence is almost additive if (6.2) holds with C_n independent of n.

gives a sequence which is asymptotically additive but not weakly almost additive. We note that these two potential sequences are actually equivalent (in the sense of Remark 0.11) to the potential which is identically zero. It remains an open question whether one can find an asymptotically additive potential $\mathcal{G}$ such that there is no additive potential in the same equivalence class.

Frequently used notation

(S)	specification property, page 18
(WPS)	weak periodic specification property, page 18
(USCE)	upper semi-continuity of entropy, page 23
(PAP)	periodic approximation of pressure, page 23
(C)	φ is continuous, page 17
(H)	φ is a homeomorphism, page 17
($\mathcal{C}$-Commutation)	commutation hypothesis, page 26
($\mathcal{R}$-Reversal)	reversal hypothesis, page 26
M	compact metric space
φ	continuous mapping of the space M into itself
M_n	set of fixed points of the mapping φ^n
$C(M)$	space of continuous functions $V : M \to \mathbb{R}$ with the supremum norm $\|\cdot\|_\infty$
$B(M)$	space of bounded measurable functions $V : M \to \mathbb{R}$ with the norm $\|\cdot\|_\infty$
$\mathcal{A}(M)$	space of asymptotically additive sequences of functions, page 7
$\mathcal{P}(M)$	set of probability measures on the space M with the Borel σ-algebra
$\mathcal{P}_\varphi(M)$	set of invariant measures for a mapping φ
$\mathcal{E}_\varphi(M)$	set of ergodic invariant measures for φ
$h_{\mathrm{Top}}(\varphi)$	topological entropy of φ
$h_\varphi(\mathbb{Q})$	Kolmogorov–Sinai entropy of φ
$\mathfrak{p}_\varphi(\mathcal{G})$	topological pressure of a continuous map φ with respect to $\mathcal{G} \in \mathcal{A}(M)$, page 20
μ_n^x	empirical measures, page 4
σ_n	the entropy production in time n, page 27

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