Holographic Hydrodynamics

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AdS/CFT and Condensed Matter workshop
Imperial College London
January 10, 2011

Holographic hydrodynamics

Expectation: low-energy effective description of interacting QFT = fluid dynamics ("hydrodynamics")

Encoded by higher-dimensional (gravitational) dual description.

- crucial role played by black holes
- rooted in AdS/CFT (gauge/gravity) correspondence

- dubbed **Fluid/Gravity correspondence...**

Invitation: fluids and GR

The idea of black holes resembling fluids is not new.

- Black hole thermodynamics

  stationary black hole horizons have temperature and entropy

  [Hawking, Bekenstein, ...]

- Analog models of black holes

  fluids can have sonic horizons

  [Unruh]

  Gregory-Laflamme instability is mimicked by Rayleigh-Plateau instability

  [Cardoso & Dias]

- The black hole Membrane Paradigm

  black holes behave as a fluid membrane with viscosity, conductivity, etc.
  with dynamics given by Navier-Stokes eqns, Ohm's law, etc.

  [Thorne, Price, Macdonald]

The Fluid/Gravity correspondence is none of these...

Preview: Fluid/Gravity correspondence

The Fluid/Gravity correspondence is a relation between fluid dynamics in (3+1) dimensions and gravity (with negative cosmological const.) in (4+1) dimensions.

For arbitrary fluid flow satisfying the (generalized) Navier-Stokes equations, we construct a solution to Einstein's equations describing a dynamical black hole in asymptotically Anti de Sitter spacetime, with regular horizon whose evolution mimics that of the fluid.

Developed within the context of gauge/gravity duality in 2008 by [Bhattacharyya, VH, Minwalla, Rangamani] building on earlier work by [Policastro, Son, Starinets; Janik, Peschanski; Bhattacharyya, Lahiri, Loganayagam, Minwalla; ...] and subsequently generalized and utilized by many groups.

Motivation:

The fluid/gravity correspondence has applications to:

* Black hole physics

  Fluid specifies a generic (evolving, non-uniform) black hole solution,
  to arbitrary accuracy in long-wavelength regime.

* Strongly coupled field theories

  Gravity determines properties of the gauge theory plasma,
  such as transport coefficients of the conformal fluid.
  This mimics physics of the quark-gluon plasma currently observed at RHIC,
  as well as certain condensed matter systems.

* Fluid dynamics

  Low-energy effective description of gauge theory is fluid dynamics.
  We can 'geometrize' long-standing fluid dynamical puzzles, such as
  understanding turbulence.

Outline:

* Background
  - highlights from gauge/gravity duality
* Solution for global equilibrium
  - Planar Schwarzschild-AdS black hole
  - Ideal fluid
* Deformations away from global equilibrium
  - Form of dissipative fluid stress tensor
  - Geometry at conceptual level: patching black holes
  - Geometry at technical level: expansion of Einstein's eqns
* General solution
  - Event horizon in the bulk geometry
  - Transport coefficients in the boundary fluid
* Further consequences of the correspondence

Background: gauge/gravity duality

The gauge/gravity duality relates strongly coupled non-abelian (SYM) gauge theory in d dimensions to string theory, which in certain regime reduces to classical gravity, on (d+1)-dimensional asymptotically Anti de Sitter spacetime.

Key aspects:

* Gravitational theory maps to non-gravitational one!
* Holographic:
  gauge theory 'lives on boundary of AdS'.
* Strong/weak coupling duality.

Background: gauge/gravity duality

Specific points:

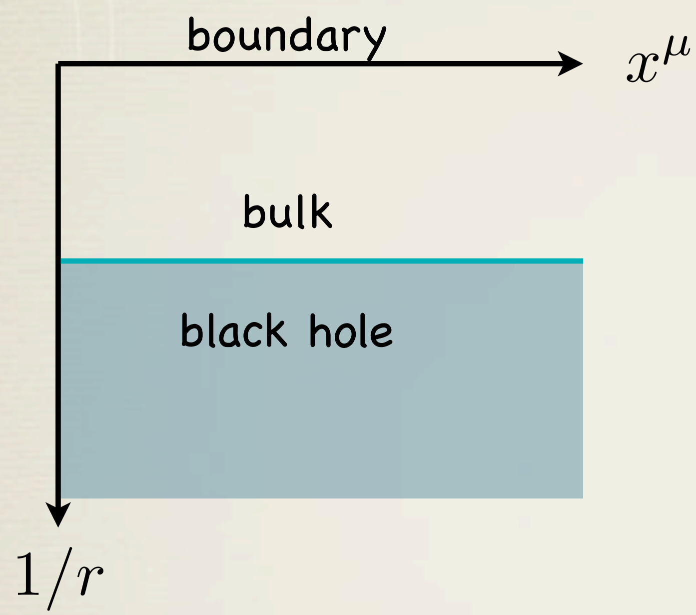
* Distinct asymptotically AdS (bulk) geometries correspond to distinct states in (boundary) gauge theory.
  * AdS bulk geometry (=maximally symmetric, negatively curved ST) corresponds to vacuum state of the gauge theory.
  * Planar Schwarzschild-AdS black hole corresponds to thermal state of the gauge theory.
  * Note: supersymmetry is not needed for this correspondence.

* Bulk geometry induces boundary stress tensor,

  which captures the essential physics of the gauge theory state (eg. local energy density, pressure, temperature, entropy current, etc.)

Background: gauge/gravity duality

Specific points:



boundary fluid specified by $T_{\mu\nu}(x^\mu)$

bulk geometry specified by $g_{ab}(r, x^\mu)$

$$ds^2 = g_{ab}\, dX^a\, dX^b \qquad X^a = \{r, x^\mu\}$$

* Bulk dynamics is specified by Einstein's equations.

$$E_{ab} \equiv R_{ab} - \frac{1}{2} R\, g_{ab} + \Lambda\, g_{ab} = 0$$

* Boundary dynamics is specified by stress tensor conservation.

$$\nabla_\mu T^{\mu\nu} = 0$$

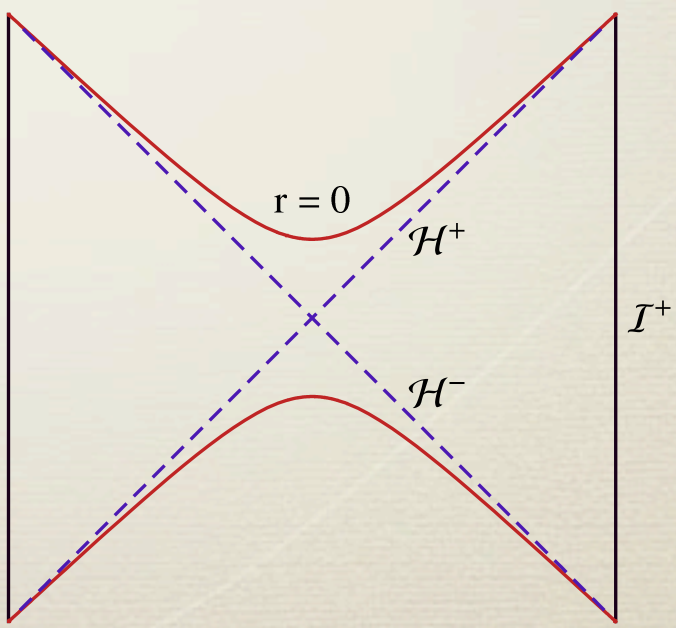
Global equilibrium:

Planar Schwarzschild-AdS$_5$ black hole metric:

$$ds^2 = r^2 \left(-f(r)\, dt^2 + \sum_i (dx^i)^2 \right) + \frac{dr^2}{r^2 f(r)} \qquad \text{with} \quad f(r) = 1 - \frac{r_+^4}{r^4}$$

BH temperature of the horizon: $T = r_+/\pi$

Causal structure:



- spacelike curvature singularity at r=0
- regular event horizon at r=r$_+$
- timelike AdS boundary at r=∞

Global equilibrium:

Planar Schwarzschild-AdS$_5$ black hole in more convenient coordinates (regular on horizon, boundary-covariant):

$$ds^2 = -2\, u_\mu\, dx^\mu dr + r^2 \left(\eta_{\mu\nu} + \frac{\pi^4 T^4}{r^4}\, u_\mu\, u_\nu \right) dx^\mu dx^\nu$$

= 4-parameter family of stationary black hole solutions, parameterized by BH temperature T and horizon velocity u^μ.

Induced boundary stress tensor:

$$T^{\mu\nu} = \pi^4\, T^4\, \left(\eta^{\mu\nu} + 4\, u^\mu\, u^\nu\right)$$

= Perfect fluid at temperature T moving with velocity u^μ (such that $u^\mu\, u_\mu = -1$).

Note: this describes a conformal fluid ($T^\mu_\mu = 0$) with no dissipation.

Fluids with dissipation

In order for the stress tensor to capture dissipation, it must allow for variations of T and u^μ.

Long wavelength regime:

In order to have a meaningful fluid description, the scale of variation L of the fluid variables T and u^μ must be large compared to the microscopic scale $1/T$.

This automatically gives us a small parameter: $$L\,T \equiv \frac{1}{\epsilon} \gg 1$$

This naturally allows us to expand $T_{\mu\nu}$ in 'boundary derivatives' $\partial_\mu(\ldots)$

Terms of order $(\partial_\mu u_\nu)^n, \ldots, \partial_\mu^n u_\nu$ will be suppressed by ϵ^n.

Fluids with dissipation

Expand stress tensor as:

$$T^{\mu\nu} = \pi^4 T^4 \left(\eta^{\mu\nu} + 4\, u^\mu\, u^\nu\right) + \Pi^{\mu\nu}_{(1)} + \Pi^{\mu\nu}_{(2)} + \ldots$$

dissipative terms composed of $\partial_\mu u_\nu$

2nd order dissipative terms

The conservation equations $\nabla_\mu T^{\mu\nu} = 0$ are the generalized Navier-Stokes equations.

The form of the stress tensor is determined by symmetries, leaving finite number of 'transport' coefficients at each order.

Fluids with dissipation

Using Weyl-covariant formalism, the form of d-dimensional dissipative stress tensor (on background $\gamma^{\mu\nu}$) to 2nd order is:

$$
\begin{aligned}
T^{\mu\nu} = \;& P\,(\gamma^{\mu\nu} + d\,u^\mu\,u^\nu) && \longrightarrow \text{0th order} \\
& -2\,\eta\,\sigma^{\mu\nu} && \longrightarrow \text{1st order} \\
& +2\,\eta\left[\tau_1\;u^\lambda \mathcal{D}_\lambda \sigma^{\mu\nu} - \tau_\in\,(\omega^\mu{}_\lambda\,\sigma^{\lambda\nu} + \omega^\nu{}_\lambda\,\sigma^{\lambda\mu})\right] && \\
& +\xi_\sigma\left[\sigma^\mu{}_\lambda\sigma^{\lambda\nu} - \tfrac{P^{\mu\nu}}{d-1}\sigma^{\alpha\beta}\sigma_{\alpha\beta}\right] + \xi_C\;C_{\mu\alpha\nu\beta}\,u^\alpha\,u^\beta && \text{2nd order} \\
& +\xi_\omega\left[\omega^\mu{}_\lambda\,\omega^{\lambda\nu} + \tfrac{P^{\mu\nu}}{d-1}\omega^{\alpha\beta}\,\omega_{\alpha\beta}\right] && \\
& +\ldots && \longrightarrow \text{higher order}
\end{aligned}
$$

where

$\sigma_{\mu\nu}$ = shear (η = shear viscosity)

$\omega_{\mu\nu}$ = vorticity

$P_{\mu\nu}$ = spatial projector

$\mathcal{D}_\mu$ = Weyl-covariant derivative

$C_{\mu\alpha\nu\beta}$ = Weyl tensor for background $\gamma^{\mu\nu}$

are quantities build out of the velocity u^μ and background metric $\gamma_{\mu\nu}$

[Loganayagam]

Fluids with dissipation

The transport coefficients depend on the microscopic structure of the fluid; they could be in principle measured, or calculated from first principles.

However, both of these approaches are prohibitively difficult, since the gauge theory is strongly coupled.

We will see that bulk gravity in fact determines the transport coefficients uniquely.

Transport coefficients can already be extracted from quasinormal modes,
i.e. small fluctuations about black holes.

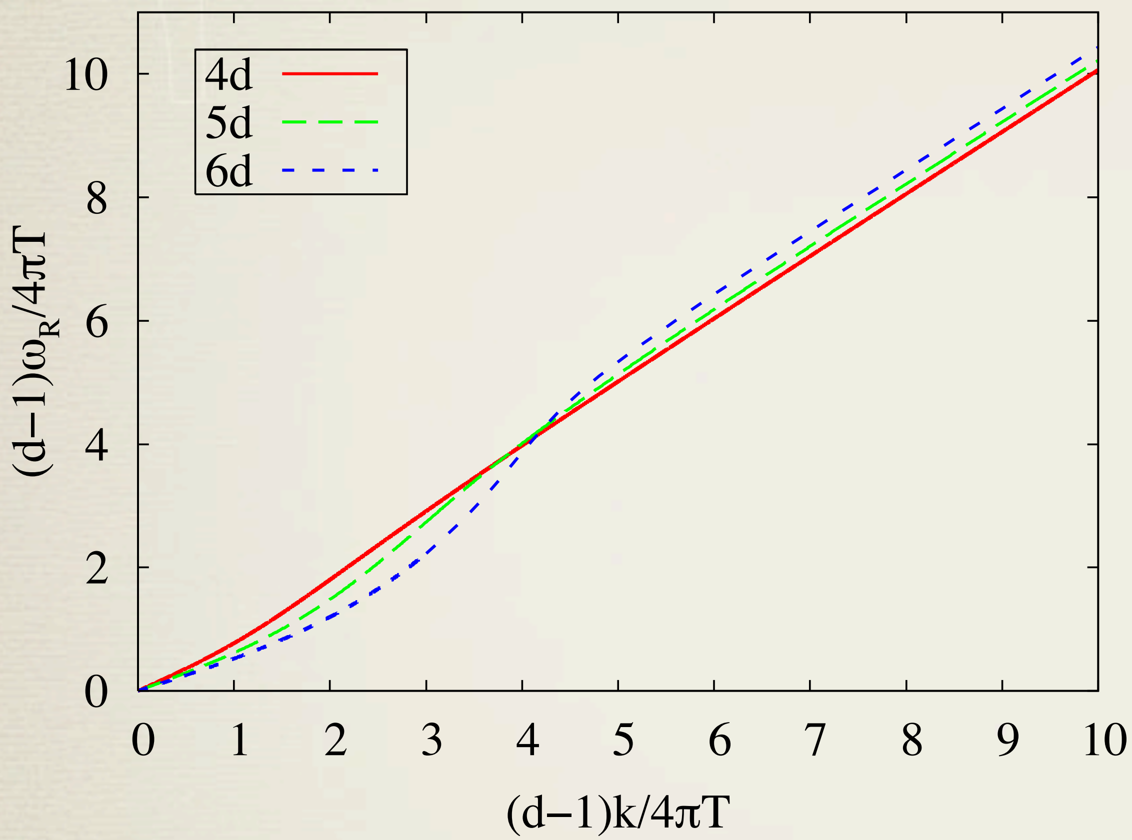
Small deviations from equilibrium:

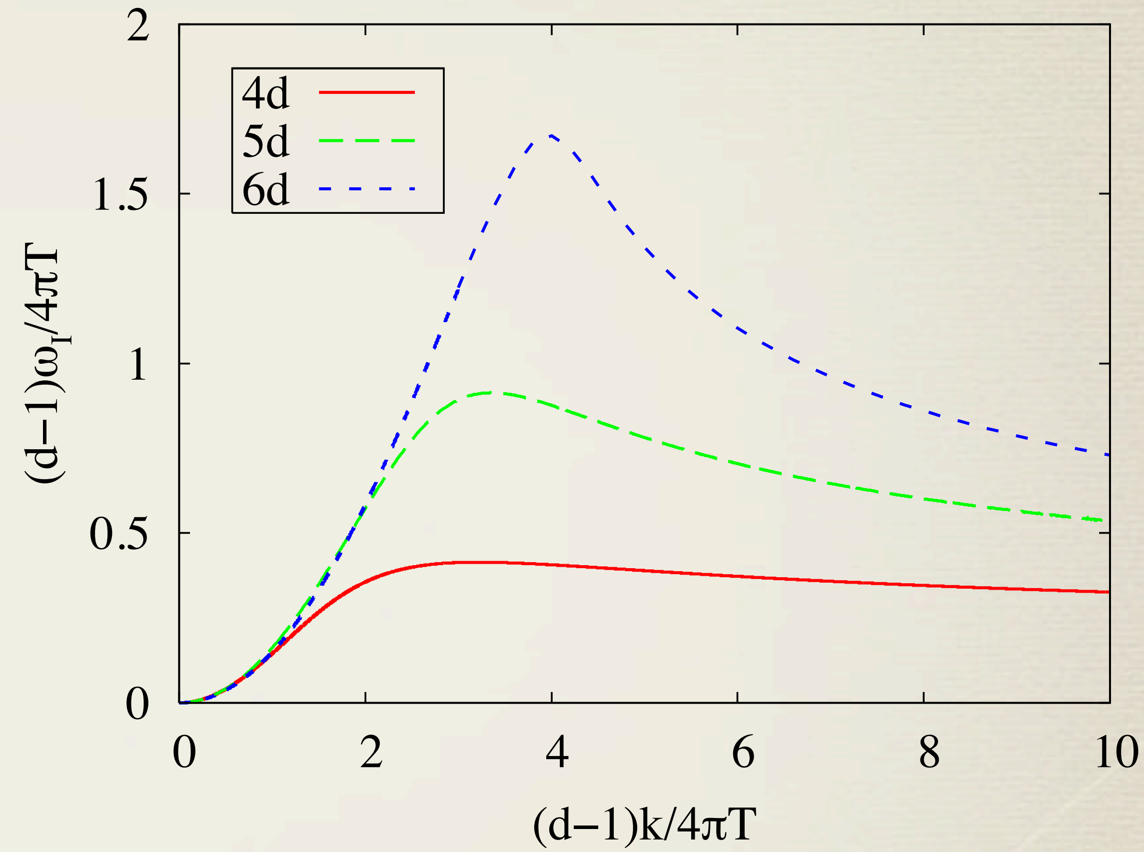
* Black hole quasinormal modes encode the field theory's return to thermal equilibrium [Horowitz, VH]
* Linear fluctuations of AdS black holes display modes with hydrodynamic dispersion relations [Policastro, Son, Starinets]
  propagating sound mode with linear dispersion and shear mode with damped quadratic dispersion
* Using linear response theory, transport coefficients can be computed [Buchel, Herzog, Kovtun, Policastro, Son, Starinets, ...]
* This led to the famous bound on shear viscosity to entropy density ratio: [Kovtun, Son, Starinets]

$$\frac{\eta}{s} \geq \frac{1}{4\pi}$$

saturated by black holes;
cold atoms at unitarity and quark-qluon plasma both come close to saturating the bound.

The sound of AdS black holes

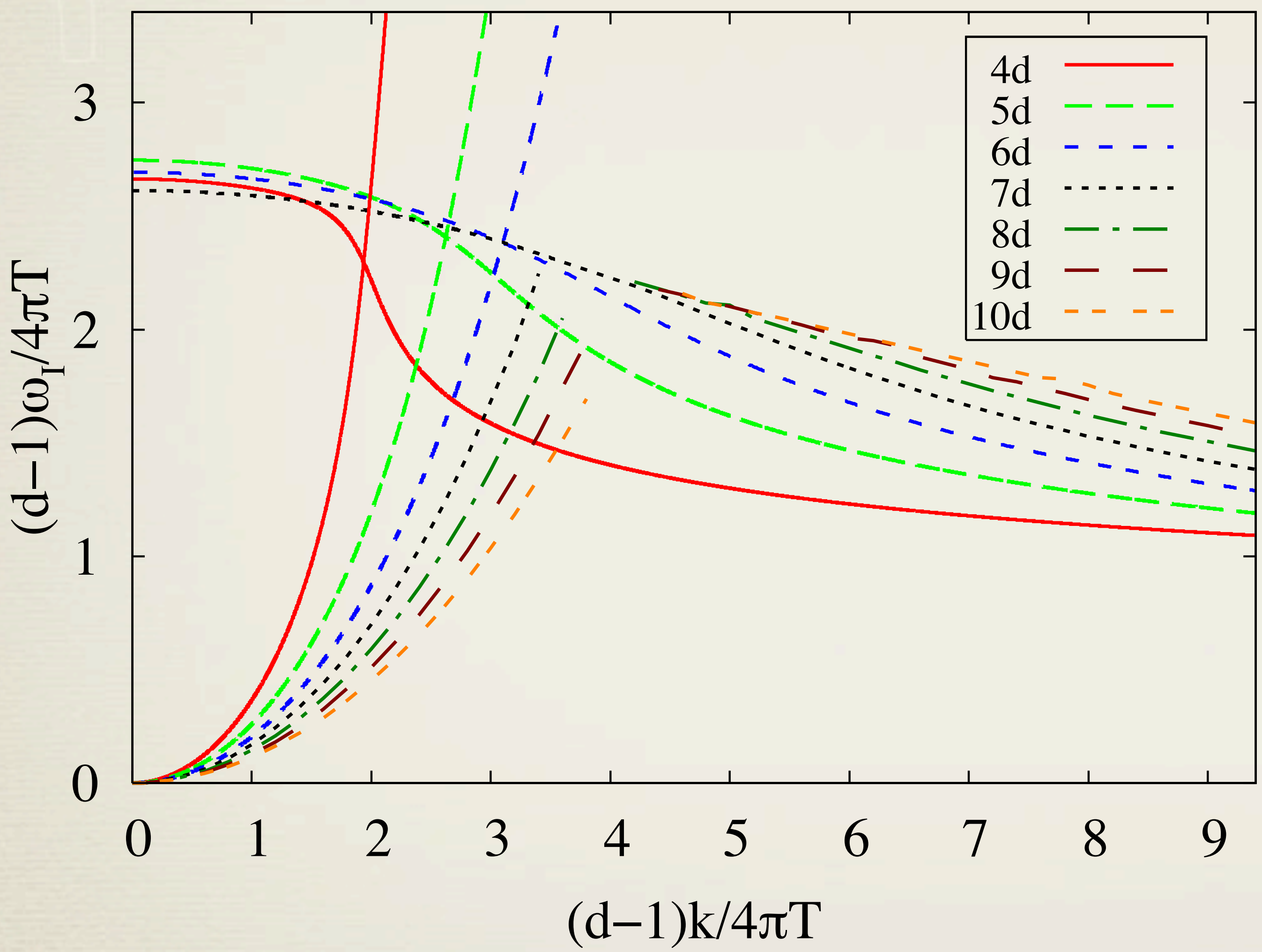




[Morgan, Cardoso, Miranda, Molino, Zanchin]

$$\omega = \frac{1}{\sqrt{d-1}}\, k - i\,\Gamma_s\, k^2$$

Shear modes of AdS black holes



[Morgan, Cardoso, Miranda, Molino, Zanchin]

$$\omega = -i\, D\, k^2$$

$$D = \frac{\eta}{d\, P}$$

$$\frac{\eta}{s} = \frac{1}{4\pi}$$

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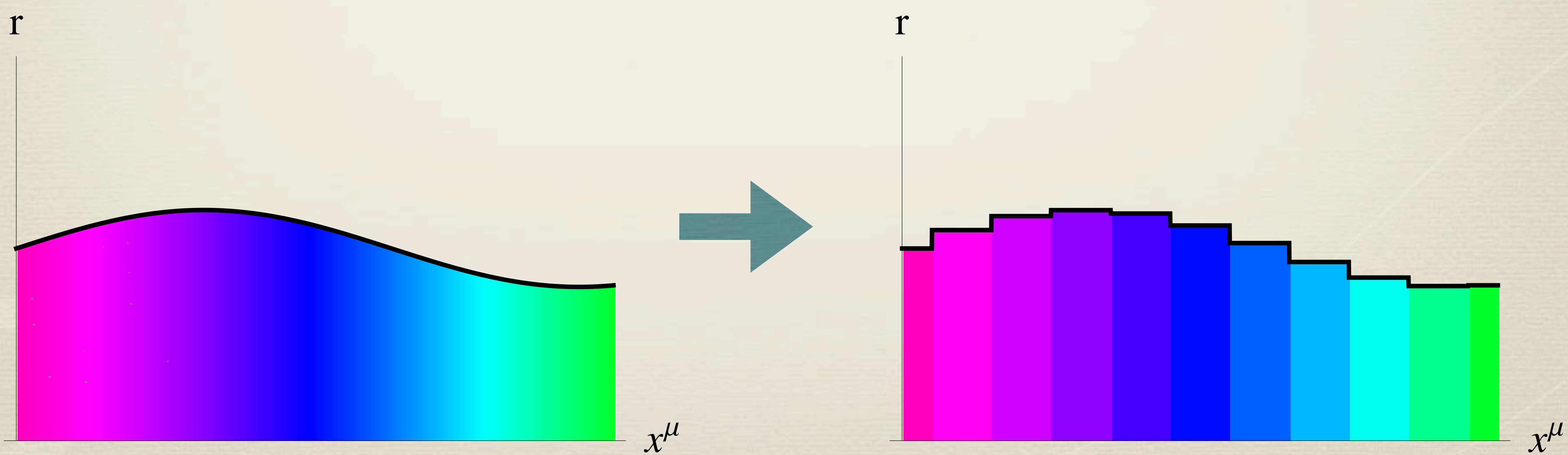
Black holes with variation

Conceptual motivation:

Suppose that the 'parameters' of black hole T and u^μ varied slowly in x^μ.

Then at each x_0^μ, the geometry should look approximately like a black hole with temperature $T(x_0)$ and velocity $u^\mu(x_0)$:



Black holes with variation

Mathematically, instead of specifying 4 parameters in

$$ds^2 = -2\, u_\mu \, dx^\mu dr + r^2 \left(\eta_{\mu\nu} + \frac{\pi^4 T^4}{r^4} u_\mu \, u_\nu \right) dx^\mu dx^\nu$$

suppose we specified the metric by 4 functions of x^μ:

$$ds^2 = -2\, u_\mu(x) \, dx^\mu dr + r^2 \left(\eta_{\mu\nu} + \frac{\pi^4 T(x)^4}{r^4} u_\mu(x) \, u_\nu(x) \right) dx^\mu dx^\nu$$

This metric is still regular (for regular $u_\mu(x)$ and $T(x)$), but does **not** solve Einstein's equations unless $u_\mu(x)$ and $T(x)$ are constant.

Iterative construction

However, we can use this as a starting point for an iterative construction:

Assuming slow variations

$$\frac{\partial_\mu \log T}{T} \sim \mathcal{O}(\epsilon) \ , \qquad \frac{\partial_\mu u}{T} \sim \mathcal{O}(\epsilon)$$

we can expand

$$g_{ab} = \sum_{k=0}^{\infty} \epsilon^k \, g_{ab}^{(k)} \ , \qquad T = \sum_{k=0}^{\infty} \epsilon^k \, T^{(k)} \ , \qquad u_\mu = \sum_{k=0}^{\infty} \epsilon^k \, u_\mu^{(k)}$$

and solve Einstein's equations order by order in ϵ.

Iterative construction

Einstein's equations

$$E_{ab} \equiv R_{ab} - \frac{1}{2} R\, g_{ab} + \Lambda\, g_{ab} = 0$$

separate into two sets:

- ✳ Constraint equations: $E_{r\mu} = 0$

$$\Longleftrightarrow \qquad \nabla_\mu T^{\mu\nu} = 0$$

  these implement stress tensor conservation

  (at one lower order)

- ✳ Dynamical equations: $E_{\mu\nu} = 0$ and $E_{rr} = 0$

  these allow us to solve for the metric correction

  (at the given order)

Iterative construction

Structure of dynamical equations:

At order $\mathcal{O}(\epsilon^k)$, having solved for all $g_{ab}^{(n)}$ with $n < k$, the dynamical equations which determine $g_{ab}^{(k)}$ take a miraculously simple form. Schematically:

$$\mathbb{H}\left[g^{(0)}(u_\mu^{(0)}, T^{(0)})\right] g^{(k)} = s_k$$

* Here $\mathbb{H}$ is a second order differential operator in the variable r alone (i.e. it contains no derivatives in x^μ).
* The RHS contains regular source terms s_k built out of the previously obtained $g_{ab}^{(n)}$ with $n < k$.

Iterative construction

Solution to the dynamical equations

$$\mathbb{H}\left[g^{(0)}(u_\mu^{(0)}, T^{(0)})\right] g^{(k)} = s_k$$

(subject to regularity in the bulk and asymptotically AdS boundary conditions)

takes the form

$$g^{(k)} = \text{particular}(s_k) + \text{homogeneous}(\mathbb{H})$$

same at all orders, ultra-local

captures higher order non-linearities

and exists provided the constraint equations are solved.

Key points of the construction

* The iterative construction can in principle be systematically implemented to arbitrary order in ϵ (which obtains correspondingly accurate solution).
* The resulting black hole spacetimes form a continuously-infinite set of (approximate) solutions: instead of specifying 4 parameters, we specify 4 functions of 4 variables.
* However, these solutions are given implicitly in terms of functions $T(x^\mu)$ and $u^\nu(x^\mu)$ which must solve the generalized Navier-Stokes equations.
* Any regular fluid dynamical solution corresponds to a bulk black hole with regular horizon.

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Bulk geometry

At **1st order** we find the black hole solution

$$ds^2 = -2\, u_\mu\, dx^\mu dr + r^2 \left(\eta_{\mu\nu} + \frac{\pi^4\, T^4}{r^4}\, u_\mu\, u_\nu\right) dx^\mu dx^\nu$$
$$+2r \left[\tfrac{r}{\pi T}\, F(r/\pi T)\, \sigma_{\mu\nu} + \tfrac{1}{3}\, u_\mu u_\nu\, \partial_\lambda u^\lambda - \tfrac{1}{2}\, u^\lambda \partial_\lambda \left(u_\nu u_\mu\right)\right] dx^\mu dx^\nu$$

where

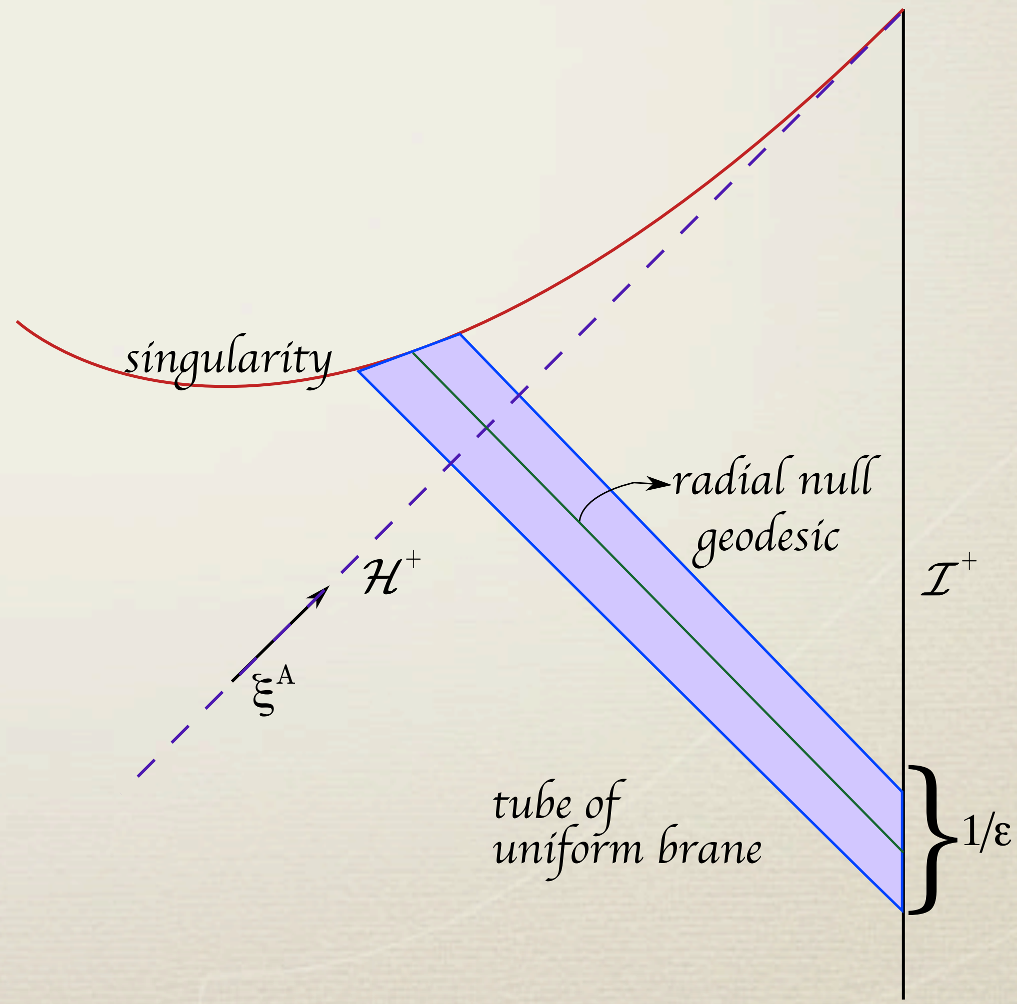
$$F(r) = \int_r^\infty dx\, \frac{x^2 + x + 1}{x(x+1)\,(x^2+1)} = \frac{1}{4}\left[\ln\left(\frac{(1+r)^2(1+r^2)}{r^4}\right) - 2\,\arctan(r) + \pi\right]$$

At **2nd order** the solution is page-long, but explicit. We can explicitly find the event horizon and confirm regularity, horizon area growth, etc.

Bulk geometry

Locally at each x^{μ}, i.e. along a radial ingoing null geodesic, the geometry approximates a uniformly boosted planar Schwarzschild-AdS black hole, with corrections suppressed by the 'speed' of variation, ϵ.

The causal structure is preserved, the horizon is regular but non-uniform and dynamically evolving.



Bulk geometry event horizon:

Assuming the dissipation causes our configuration to settle down to a stationary state at late times, we can find the event horizon as the unique null hypersurface with the correct late-time behavior.

This can be solved algebraically, order-by-order in ϵ.

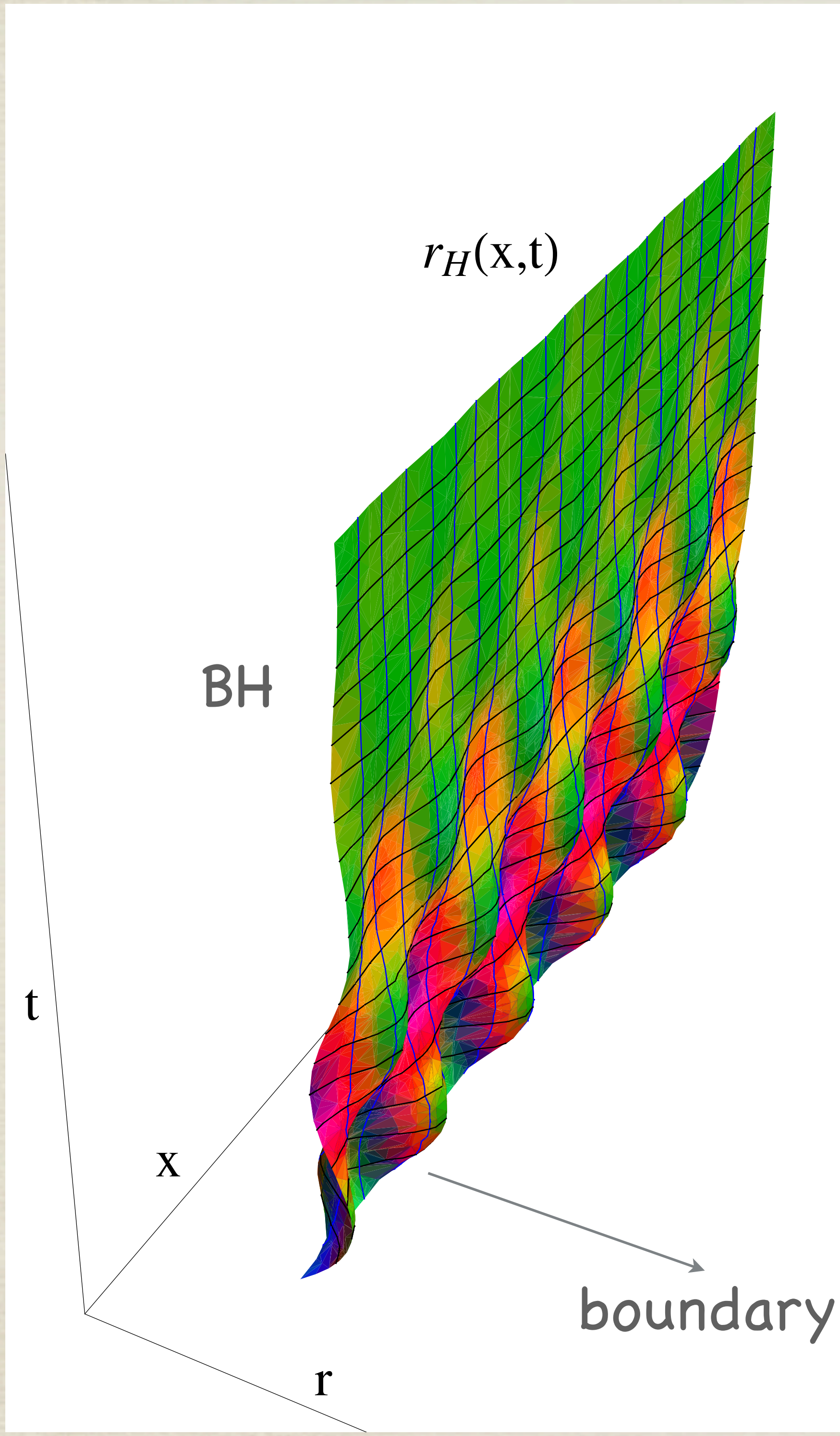
$$r_+(x) = \pi\, T(x) + \frac{1}{\pi\, T(x)} \left(\#\, \sigma_{\mu\nu}\, \sigma^{\mu\nu}(x) + \#\, \omega_{\mu\nu}\, \omega^{\mu\nu}(x) \right) + \ldots$$

The event horizon is determined locally at a given x^μ !

(Though unusual, this is possible due to the long wavelength regime, i.e. horizon position varies sufficiently slowly.)

[Bhattacharyya, VH, Loganayagam, Mandal, Minwalla, Morita, Rangamani, Reall]

Cartoon of the event horizon:



- Initially non-uniform horizon evolves.
- The horizon area grows.
- At late times, the horizon settles down to stationary configuration.
- The pull-back of the area form on the horizon provides a natural entropy current in the dual fluid.
- Such entropy current automatically satisfies the 2nd Law of thermodynamics.

Boundary stress tensor at 1st order

The 1st order solution induces on the AdS boundary the stress tensor of the expected form:

$$T^{\mu\nu} = \pi^4 T^4 (4\, u^\mu u^\nu + \eta^{\mu\nu}) - 2\,\pi^3 T^3\, \sigma^{\mu\nu}$$

From this we can read off the shear viscosity of the fluid; in terms of shear viscosity to entropy density ratio (calculated previously [Policastro, Son, Starinets]),

$$\frac{\eta}{s} = \frac{1}{4\pi}$$

This saturates the famous lower bound conjectured by [Kovtun, Son, Starinets].

Boundary stress tensor at 2nd order

Transport coefficients in 2nd order stress tensor in d dimensions:

$$P = \frac{1}{16\pi G_{\text{AdS}}} \left(\frac{4\pi T}{d}\right)^{d}$$

$$\eta = \frac{s}{4\pi} = \frac{1}{16\pi G_{\text{AdS}}} \left(\frac{4\pi T}{d}\right)^{d-1}$$

$$\tau_1 = \frac{d}{4\pi T} \left(1 - \int_1^\infty dy\, \frac{y^{d-2} - 1}{y(y^d - 1)}\right)$$

$$\tau_\epsilon = \frac{d}{4\pi T} \int_1^\infty dy\, \frac{y^{d-2} - 1}{y(y^d - 1)}$$

$$\xi_\sigma = \xi_C = \frac{d}{2\pi T}\, \eta$$

$$\xi_\omega = 0$$

- concurrently derived in d=4 by [Baier, Romatschke, Son, Starinets, Stephanov]
- higher dimensions: [Haack, Yarom; Bhattacharyya, Loganayagam, Mandal, Minwalla, Sharma]

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  - Transport coefficients in the boundary fluid
* Generalizations and further consequences

Generalizations

* **Different number of dimensions**

  Closed form expressions for 2nd order solution exist [Bhattacharyya et.al., Haack et.al.]
  Intriguing observation [van Raamsdonk]: For incompressible fluids, 2+1 dimensional dynamics is qualitatively different from 3+1 dimensional dynamics (eg. inverse energy cascade). What are the implications for gravity in 4 vs. 5 dimensions?
* **Fluids on curved background** [Bhattacharyya et.al.]
* **Adding other bulk matter fields** [Bhattacharyya et.al., Erdmenger et.al.]

  e.g. Einstein-Maxwell, Einstein-Dilaton, etc.
  holographic superfluid (charged scalar) [Sonner & Withers]
  These introduce richer physics, at expense of universality.
  Adding bulk gauge fields gives new local conserved charge.
  Adding bulk dilaton induces forcing of the fluid.
* **Zero-temperature fluids** [Hansen et.al., Erdmenger et.al., Banerjee et.al., Oh]
* **Non-conformal fluid dynamics** [Kanitscheider et.al., David et.al]
* **Non-relativistic fluid dynamics** [Bhattacharyya et.al., Rangamani et.al.]

Implications for GR & fluid dynamics

- **Improvement on Israel-Stewart formalism**

  At first order, relativistic viscous fluid is described by parabolic system, which leads to apparent causality violations. Israel-Stewart formalism renders the system hyperbolic by adding some 2nd order terms, but not all. Fluid/gravity construction prescribes the correct completion to render the system causal.

- **New contribution to charge current**

  For Maxwell-Chern-Simons charged fluid, we find a surprise at 1st order: in addition to standard dissipative terms, a new (parity-violating but CP preserving) term appears in charge current: $\ell^\mu = \epsilon_{\alpha\beta\gamma}{}^{\mu}\, u^\alpha \, \nabla^\beta u^\gamma$

  [Erdmenger et.al., Banerjee et.al., Son et.al.]

  This term has been ignored by Landau&Lifshitz, but it may have potentially observable effects.

- **Blackfold approach to constructing higher-dim BHs**

  [Emparan et.al.]

- **Convenient rewriting of rotating AdS black holes in terms of fluid variables** [Bhattacharyya et.al.]

Implications for condensed matter physics

Bad news: fluid/gravity is not going to solve your favorite CM system:

Fluids one encounters in every-day experience do not have a dual in terms of a classical gravitational system (typically not enough DoFs).

What types of fluids do have gravitational dual?

Conjecture: [Heemskerk, Penedones, Polchinski, Sully]

Any CFT that has a planar expansion, and in which all single-trace operators of spin greater than two have parametrically large dimensions, has a local bulk dual.

Implications for condensed matter physics

Fluid/gravity correspondence is most useful for elucidating general guiding principles and universal features.

(BH captures universal dynamics of stress tensor of any CFT with grav. dual)

Fluid/gravity moreover provides a handle on nonlinear departures from equilibrium.

Good news: fluid/gravity can suggest what sorts of processes might be possible & when to expect them.

It simultaneously provides a new (geometrical) perspective on many familiar hydrodynamic phenomena.

Summary:

* Fluid/gravity correspondence maps asymptotically AdS black hole dynamics to lower-dimensional dynamics of fluids.

* Fluid/gravity correspondence is a rapidly-expanding area, and provides a useful tool for
  * studying behavior of generic black hole horizon
  * geometrizing fluid dynamics
  * gaining insight into behavior of strongly coupled field theories, which exhibit similar features to certain real-world systems.