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# Introduction to Relativistic Transport Theory

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# Preface

The present manuscript are lecture notes on an introduction to relativistic transport theory, held at the CNT Lectures on Hot/Dense Matter 2015, Variable Energy Cyclogron Center in Kolkata, India.

**Notation and conventions:** In these notes we shall use a “natural system of units” as usually applied in theoretical high-energy particle and nuclear physics, i.e., we set the modified Planck constant  $\hbar = 1$  (except in places, where we want to emphasize the statistical measure for phase space) and the speed of light  $c = 1$ . This implies that masses, energy and momenta are measured in, e.g., GeV and times and lengths in  $\text{GeV}^{-1}$ . Sometimes it is also convenient to measure times  $ct$  and lengths in fm ( $1 \text{ fm} = 10^{-15} \text{ m}$ ). Then the conversion factor  $\hbar c = 0.197 \text{ GeVfm}$  is used. Further we use the “west-coast convention” concerning the Minkowski pseudo-metric,  $\eta_{\mu\nu} = \eta^{\mu\nu} = \text{diag}(1, -1, -1, -1)$ . Masses of particles always denote the invariant (or “rest”) mass of the particles. Local macroscopic quantities like densities, temperatures, etc. always refer to the local rest frame of the “fluid cells” and thus are Minkowski scalars.

## Literature:

For (semi-)classical kinetic theory: A very good general introduction into both classical and quantum transport theory, including a lot of applications, however (mostly) restricted to the non-relativistic theory is [LP81]. For the relativistic theory I used [CK02] and the first chapters on the classical theory of [dvv80].

For quantum-field theoretical approaches to quantum-transport theory in terms of the Schwinger-Keldysh real-time formulation of many-body relativistic quantum field theory the seminal papers are [Sch61, Kel64]. A very good introduction to the non-relativistic theory is [Dan84a, Dan84b]. A standard review reference, particularly with applications to nuclear physics is [BM90]. For a recent review on a modern transport-code realization for nuclear-physics applications<sup>1</sup> see [BGG<sup>+</sup>12].

The seminal papers on the  $\Phi$ -functional approach to the quantum description of many-body systems are [BK61, Bay62]. In these lectures we refer to [Cas09, KIV01]. For my manuscript on relativistic equilibrium and nonequilibrium quantum-field theory, see [Hee13].

Some standard textbooks on non-relativistic many-body theory are [FW71, KB61, AS10, Ram07] and for relativistic many-body theory in thermal equilibrium [LeB96, KG06].

Frankfurt, Spring 2015,  
Hendrik van Hees.

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<sup>1</sup>The Giessen Boltzmann-Uehling-Uhlenbeck Project, GiBUU

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## Chapter 1

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# Classical Relativistic Transport Equations

In this Chapter we shall establish the basic ideas about kinetic theory for classical relativistic particles. We start with a brief reminder on the special theory of relativity and then discuss the covariant notion of phase-space distribution function and derive a relativistic Boltzmann equation, including elastic two-body collisions. After this we discuss the validity of conservation laws, the  $H$  Theorem and derive the distribution for local and global thermal equilibrium. We conclude this chapter with an extension of these concepts to take into account quantum-statistical effects for the kinetic description of degenerate relativistic gases (Boltzmann-Uehling-Uhlenbeck equation).

### 1.1 Reminder on Special Relativity

The special theory of relativity originates from the problem that the Maxwell Equations of classical electromagnetism are not invariant under Galilei transformations as is Newtonian mechanics. This implies that there should be a preferred frame of reference, which was associated with the rest frame of a hypothetical medium, called the ether. On the other hand, its existence could never be established experimentally. After some previous work by FitzGerald, Lorentz, Poincaré, H. Hertz and others, Einstein came to the conclusion that the description of space and time has to be modified, which affects not only the theory of electromagnetic phenomena but all physics. Here, we will summarize the special theory of relativity in modern covariant notation. We adapt the convention that we set the **speed of light**, which is a fundamental natural constant according to relativistic physics,  $c = 1$ . Further, following Minkowski, it is convenient to combine the time and the Cartesian coordinates of the position vector, describing locations in space as appearing to an observer who is at rest with respect to an inertial frame of reference to the four-vector

$$x = (x^\mu) = \begin{pmatrix} t \\ \vec{x} \end{pmatrix} = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}. \quad (1.1.1)$$

Here  $\mu \in \{0, 1, 2, 3\}$  is an index, labeling the components of this four-vector.

The space-time structure is further determined by the fact that the speed of light is independent of the velocity of the light source with respect to any inertial frame. Consequently, the transformation that describes the space-time coordinates of another inertial frame, where the four-vector has components  $x'^\mu$ , moving with constant velocity relative to the original inertial frame must fulfill the condition

$$t^2 - \vec{x}^2 = t'^2 - x'^2 - y'^2 - z'^2 = t'^2 - \vec{x}'^2 = t'^2 - x'^2 - y'^2 - z'^2. \quad (1.1.2)$$

For a light signal, that is sent from the origin of the coordinate system  $\vec{x}' = \vec{x} = 0$  at time  $t = t' = 0$  that as reached coordinates  $\vec{x}$  at time  $t$  or  $\vec{x}'$  at time  $t'$  with respect to the old and new inertial frames, respectively, this means that  $t^2 - \vec{x}^2 = t'^2 - \vec{x}'^2 = 0$ , i.e., in both coordinate systems, the light moves with speed  $c = 1$ .

From the perspective of four-dimensional linear algebra that means that the  $4 \times 4$ -transformation matrix  $\Lambda^\mu{}_\nu$  has to fulfill certain constraints, which we shall derive now. The transformation from the four-vector coordinates of the old inertial frame to the coordinates of the new one read

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu, \quad (1.1.3)$$

where we adopt the **Einstein summation convention**, according to which one has to sum over equally named indices, where one has to be written as a superscript and the other as a subscript. We shall come back to this convention of upper and lower indices in a moment. The quadratic form (1.1.2) can be written as

$$x \cdot x = t^2 - \vec{x}^2 = \eta_{\mu\nu} x^\mu x^\nu, \quad (1.1.4)$$

where

$$(\eta_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \text{diag}(1, -1, -1, -1). \quad (1.1.5)$$

We can generalize this quadratic form immediately to the bilinear form

$$x \cdot y = \eta_{\mu\nu} x^\mu y^\nu = t_x t_y - \vec{x} \cdot \vec{y}. \quad (1.1.6)$$

This bilinear form is called **Minkowski product**. It has all the properties of a usual scalar product except that it is not positive definite.

Now the invariance of the quadratic form, cf. (1.1.2), should hold for all vectors, which implies

$$(x + y) \cdot (x + y) = x \cdot x + y \cdot y + 2x \cdot y \stackrel{!}{=} (x' + y') \cdot (x' + y') = x' \cdot x' + y' \cdot y' + 2x' \cdot y'. \quad (1.1.7)$$

Since  $x \cdot x = x' \cdot x'$  and  $y \cdot y = y' \cdot y'$  this implies that also

$$x \cdot y = x' \cdot y' \quad (1.1.8)$$

Using the transformation law (1.1.3), this means that

$$x \cdot y = \eta_{\mu\nu} x'^\mu y'^\nu = \eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma x^\rho y^\sigma \stackrel{!}{=} \eta_{\rho\sigma} x^\rho y^\sigma = x \cdot y \quad (1.1.9)$$

must hold for all  $x, y \in \mathbb{R}^4$ , and this implies that

$$\eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}. \quad (1.1.10)$$

Any  $4 \times 4$  matrix that obeys this condition is called a **Lorentz-transformation matrix**. To see the explicit form of a Lorentz transformation, we consider the case of a **rotation free boost** along the  $x$ -axis, i.e., the new reference frame is assumed to move with constant speed along the  $x$ -direction with the coordinate system in the same direction as the old one. This implies that  $y' = y$  and  $z' = z$ , which means that this special Lorentz-transformation matrix must take the form

$$\hat{\Lambda} = (\Lambda^\mu{}_\nu) = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (1.1.11)$$



1.1 · Reminder on Special Relativity

The condition (1.1.10) can be written in matrix-product notation

$$\hat{\Lambda} \hat{\eta} \hat{\Lambda}^T = \hat{\eta}, \quad (1.1.12)$$

which means

$$a^2 - c^2 = 1, \quad ab - cd = 0, \quad b^2 - d^2 = -1. \quad (1.1.13)$$

Obviously we can fulfill the first equation by setting

$$a = \cosh \eta, \quad c = -\sinh \eta, \quad \eta \in \mathbb{R}. \quad (1.1.14)$$

From the second equation in (1.1.13) we find

$$b = \frac{cd}{a} = -d \tanh \eta. \quad (1.1.15)$$

Plugged this into the third equation in (1.1.13), we get

$$d^2(1 - \tanh^2 \eta) = 1. \quad (1.1.16)$$

Now

$$1 - \tanh^2 \eta = 1 - \frac{\sinh^2 \eta}{\cosh^2 \eta} = \frac{1}{\cosh^2 \eta}. \quad (1.1.17)$$

Using this in (1.1.16) we find

$$d = \pm \cosh \eta. \quad (1.1.18)$$

and with (1.1.15)

$$b = -d \tanh \eta = \mp \sinh \eta. \quad (1.1.19)$$

Thus our transformation so far reads

$$t' = t \cosh \eta - x \sinh \eta, \quad x' = \pm(-t \sinh \eta + x \cosh \eta). \quad (1.1.20)$$

The velocity of the origin of the new reference frame is given by  $x' = 0 \Leftrightarrow x = vt$ , i.e.,

$$v = \frac{\sinh \eta}{\cosh \eta} = \tanh \eta. \quad (1.1.21)$$

From

$$\cosh \eta = \frac{1}{\sqrt{1 - \tanh^2 \eta}} = \frac{1}{\sqrt{1 - v^2}}, \quad \sinh \eta = \cosh \eta \tanh \eta = \frac{v}{\sqrt{1 - v^2}} \quad (1.1.22)$$

we get

$$x' = \pm \frac{1}{\sqrt{1 - v^2}}(x - vt). \quad (1.1.23)$$

For  $|v| \ll 1$  we find (up to corrections of order  $v^2$ ) the **Galilei transformation**, if we choose the upper sign. Thus the desired Lorentz transformation is given by

$$t' = \gamma_v(t - vx), \quad x' = \gamma_v(x - vt), \quad y' = y, \quad z' = z \quad \text{with} \quad \gamma_v = \frac{1}{\sqrt{1 - v^2}}. \quad (1.1.24)$$

From this derivation we see that the relative velocity between two inertial frames can never exceed the speed of light, because from (1.1.24) we must have  $|v| < 1$  to guarantee that with  $t$  and  $x$  also  $t'$  and  $x'$  are real time and space components of a real four-vector.

Next we consider the **kinematics of a classical point particle**. To find **covariant quantities** we must aim to formulate all the equations in terms of **four-vectors or tensors and scalars**. As in Newtonian mechanics we can describe the motion of the particle by giving its position as a function of time,  $\vec{x} = \vec{x}(t)$ . If we want to describe the velocity of the particle, it is given by

$$\vec{w} = \frac{d\vec{x}}{dt}. \quad (1.1.25)$$

However, under Lorentz transformations the time  $t$  is not invariant, and thus the velocity in the new reference frame looks pretty complicated.

Fortunately we can form an invariant expression with the four-vector components of the four-vector increment  $dx$ :

$$dx \cdot dx = dt^2 \left[ \left( \frac{dt}{dt} \right)^2 - \left( \frac{d\vec{x}}{dt} \right)^2 \right] = dt^2 (1 - \vec{w}^2). \quad (1.1.26)$$

Since the velocity of a particle cannot exceed the speed of light, we can define the real quantity

$$d\tau = dt \sqrt{1 - \vec{w}^2}, \quad (1.1.27)$$

which is called **proper time of the particle** under consideration. In an inertial frame, where the particle is momentarily at rest, we have  $d\tau = dt$ . This means that  $d\tau$  is the time increment measured by an observer which is momentarily at rest relative to the particle. From its derivation we see that this is a Lorentz-invariant quantity, because

$$d\tau^2 = dx \cdot dx = dx' \cdot dx' = dt'^2 (1 - \vec{w}'^2) = d\tau'^2 \quad (1.1.28)$$

holds under arbitrary Lorentz transformations of the space-time vectors. Now we can define the four-velocity of the particle by

$$u^\mu = \frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dt} \frac{dt}{d\tau} = \frac{1}{\sqrt{1 - \vec{w}^2}} \begin{pmatrix} 1 \\ \vec{w} \end{pmatrix}. \quad (1.1.29)$$

This quantity obviously transforms as a four-vector, i.e.,

$$u'^\mu = \frac{dx'^\mu}{d\tau} = \frac{1}{\sqrt{1 - \vec{w}'^2}} \begin{pmatrix} 1 \\ \vec{w}' \end{pmatrix} = \Lambda^\mu{}_\nu u^\nu. \quad (1.1.30)$$

Plugging in the above result for the Lorentz-boost matrix we get

$$u'^0 = \frac{1}{\sqrt{1 - v^2}} (u^0 - u^1 v), \quad u'^1 = \frac{1}{\sqrt{1 - v^2}} (u^1 - v u^0), \quad u'^2 = u^2, \quad u'^3 = u^3. \quad (1.1.31)$$

After some algebra this gives

$$\vec{w}' = \frac{\vec{u}'}{u'^0} = \frac{1}{1 - v w^1} \begin{pmatrix} w^1 - v \\ w^2 / \gamma_v \\ w^3 / \gamma_v \end{pmatrix}. \quad (1.1.32)$$

## 1.1 · Reminder on Special Relativity

The energy-momentum four-vector is defined as

$$p^\mu = m u^\mu = \frac{m}{\sqrt{1-\vec{w}^2}} \begin{pmatrix} 1 \\ \vec{w} \end{pmatrix}, \quad (1.1.33)$$

where  $m$  is the **invariant mass** of the particle<sup>1</sup>. For  $|\vec{w}| \ll 1$  we can expand the square root to get

$$E := p^0 = m + \frac{m}{2} \vec{w}^2 + \mathcal{O}(|\vec{w}|^4), \quad \vec{p} = m \vec{w} + \mathcal{O}(|\vec{w}|^3). \quad (1.1.34)$$

In this limit we find the Newtonian values for the kinetic energy (up to a constant shift of the energy by the invariant mass of the particle!) and momentum of a particle. From (1.1.33) we find the **energy-momentum relation** for a relativistic particle,

$$E = \sqrt{m^2 + \vec{p}^2}. \quad (1.1.35)$$

We can derive this by taking the invariant Minkowski product of the four-momentum vector,

$$p \cdot p = m^2 u \cdot u = m^2 \eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = m^2. \quad (1.1.36)$$

Splitting in temporal and spatial components, we obtain

$$(p^0)^2 - \vec{p}^2 = E^2 - \vec{p}^2 = m^2, \quad (1.1.37)$$

which immediately leads back to (1.1.35).

The equations of motion for a particle can again be guessed by generalizing the Newtonian equation  $\vec{F} = d\vec{p}/dt$  in a covariant way. A four-vector, related to the time derivative of the momentum obviously is  $dp^\mu/d\tau$ . Thus we write the **covariant equation of motion** in the form

$$\frac{dp^\mu}{d\tau} = K^\mu, \quad (1.1.38)$$

where  $K^\mu$  is the **Minkowski-four-force vector**. We note that the components of this vector can not be independent of each other, because

$$p \cdot p = m^2 = \text{const} \Rightarrow p \cdot \frac{dp}{d\tau} = 0, \quad (1.1.39)$$

which implies

$$p \cdot K = \eta_{\mu\nu} p^\mu K^\nu = 0. \quad (1.1.40)$$

As an important example we give the force on a particle with charge  $q$  in an external electromagnetic field. In relativistic notation the electromagnetic field is described via the four-potential  $A^\mu$  by the antisymmetric **Faraday tensor**

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \quad (1.1.41)$$

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<sup>1</sup>In the older literature one sometimes finds a “relativistic mass”, defined by  $m/\sqrt{1-w^2}$ , which is not a Lorentz-covariant quantity, and which we thus do not use in these lectures!

To make contact with the usual 1 + 3-dimensional notation of the electromagnetic field in terms of  $\vec{E}$  and  $\vec{B}$  we look at the corresponding temporal-spatial and spatial-spatial components of this second-rank tensor field<sup>2</sup>:

$$F^{0j} = -F^{j0} = \dot{A}^j + \partial_j A^0 = -E^j, \quad F^{jk} = -\partial_j A^k + \partial_k A^j = \epsilon^{kjl} B^l. \quad (1.1.42)$$

Here we have used that  $\partial^j = -\partial_j = -\partial/\partial x^j$  and

$$\vec{E} = -\partial_t \vec{A} - \vec{\nabla} A^0, \quad \vec{B} = \vec{\nabla} \times \vec{A}. \quad (1.1.43)$$

The four-vector (Minkowski) force for a particle in an external electromagnetic field is then given

$$K^\mu = q F^{\mu\nu} \frac{dx_\nu}{d\tau} = q F^{\mu\nu} u_\nu. \quad (1.1.44)$$

Since  $F^{\mu\nu}$  builds the contravariant components of a second-rank Minkowski tensor, this is obviously a four-vector as it should be. Further since  $p_\mu = m u_\nu$ , we have

$$p_\mu K^\mu = q F^{\mu\nu} \frac{p_\mu p_\nu}{m} = 0, \quad (1.1.45)$$

because  $F_{\mu\nu} = -F_{\nu\mu}$ . Thus the constraint (1.1.40) is fulfilled, and one of the four equations of motion (1.1.38) is redundant. Thus it is sufficient to solve for the three spatial components. For these we find a more familiar form by using (1.1.42) and writing out the tensor-vector product on the right-hand side of (1.1.44):

$$\begin{aligned} K^j &= q F^{j\nu} u_\nu = q(F^{j0} u_0 + F^{jk} u_k) = q(E^j u^0 - \epsilon^{kjl} B^l u^k) \\ &= q(E^j u^0 + \epsilon^{jkl} u^k B^l) = q u^0 E^j + q(\vec{u} \times \vec{B})^j. \end{aligned} \quad (1.1.46)$$

Now we have  $u^0 = \gamma_v$  and  $u^\mu = \gamma_v(1, \vec{v})$ . This leads to

$$\vec{K} = \gamma_v q(\vec{E} + \vec{v} \times \vec{B}) = \gamma_v \vec{F}. \quad (1.1.47)$$

Thus we can write the spatial components of the equations of motion (1.1.38) in the form

$$\frac{d\vec{p}}{d\tau} = \gamma_v \frac{d\vec{p}}{dt} = \gamma_v q(\vec{E} + \vec{v} \times \vec{B}), \quad (1.1.48)$$

which results in the three-dimensional (*not covariant!*) form

$$\frac{d\vec{p}}{dt} = q(\vec{E} + \vec{v} \times \vec{B}). \quad (1.1.49)$$

This is, although not manifestly covariant, a fully valid relativistic equation of motion. On the right-hand side we have the well-known **Lorentz force** on a point particle in an external electromagnetic field  $(\vec{E}, \vec{B})$ . As we see, usually  $\vec{F}$  is a function of both  $\vec{x}$  and  $\vec{v}$  (or in covariant form  $\vec{K}$  a function of  $\vec{x}$  and  $\vec{p}$ ).

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<sup>2</sup>In this manuscript we use the usual rationalized Gaussian units (Heaviside-Lorentz units) in electromagnetism, where the unit of charge is defined such that the static Coulomb force is  $F = q_1 q_2 / (4\pi r^2)$ , as is the common modern convention in theoretical high-energy physics (QED).

1.1 · Reminder on Special Relativity

We close this Section with a brief review of covariant vector calculus. A **scalar field** is defined as a function  $\phi(x) = \phi(t, \vec{x})$  which transforms under Lorentz transformations according to the rule

$$\phi'(x') = \phi(x) = \phi(\Lambda^{-1}x'). \quad (1.1.50)$$

Now we want to see, how the **four-dimensional generalization of the gradient** transforms under Lorentz transformations. To this end we simply use the chain rule for partial derivatives,

$$\frac{\partial \phi'(x')}{\partial x'^{\mu}} = \frac{\partial \phi(x)}{\partial x'^{\mu}} = \frac{\partial \phi(x)}{\partial x^{\nu}} \frac{\partial x^{\nu}}{\partial x'^{\mu}}. \quad (1.1.51)$$

According to our summation convention the four-gradient should have a lower index, i.e.,

$$\partial_{\nu} \phi(x) := \frac{\partial \phi(x)}{\partial x^{\nu}}. \quad (1.1.52)$$

Now we have

$$x' = \hat{\Lambda}x \Rightarrow x = \hat{\Lambda}^{-1}x'. \quad (1.1.53)$$

This gives

$$\frac{\partial x^{\nu}}{\partial x'_{\mu}} = \partial'_{\mu} x^{\nu} = (\Lambda^{-1})^{\nu}_{\mu}. \quad (1.1.54)$$

Now from (1.1.12) we find, because of  $\hat{\eta}^{-1} = \hat{\eta}$ ,

$$\hat{\Lambda}^{-1} = \hat{\eta} \hat{\Lambda}^T \hat{\eta}. \quad (1.1.55)$$

To write this in index notation, we need to define

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \text{diag}(1, -1, -1, -1). \quad (1.1.56)$$

Then we have

$$(\Lambda^{-1})^{\nu}_{\mu} = \eta^{\nu\sigma} \eta_{\mu\rho} \Lambda^{\rho}_{\sigma}. \quad (1.1.57)$$

This implies that

$$\partial'_{\mu} \phi' = (\Lambda^{-1})^{\nu}_{\mu} \partial_{\nu} \phi = \eta^{\nu\sigma} \eta_{\mu\rho} \Lambda^{\rho}_{\sigma} \partial_{\nu} \phi. \quad (1.1.58)$$

Contracting this with  $\eta^{\mu\alpha}$  we find from  $\eta^{\mu\alpha} \eta_{\mu\rho} = \delta^{\alpha}_{\rho}$ , with the Kronecker symbol

$$\delta^{\mu}_{\rho} = \begin{cases} 1 & \text{if } \rho = \mu, \\ 0 & \text{if } \rho \neq \mu, \end{cases} \quad (1.1.59)$$

that

$$\eta^{\mu\alpha} \partial'_{\mu} \phi' = \eta^{\nu\sigma} \delta^{\alpha}_{\rho} \Lambda^{\rho}_{\sigma} \partial_{\nu} \phi = \Lambda^{\alpha}_{\sigma} (\eta^{\nu\sigma} \partial_{\nu} \phi). \quad (1.1.60)$$

This means that the expression

$$\partial^{\alpha} \phi := \eta^{\alpha\mu} \partial_{\mu} \phi \quad (1.1.61)$$

transforms as four-vector components with upper indices. One says that components like  $\partial_{\mu} \phi$  with a lower index, which transforms according to the rule (1.1.58), transform **contragrediently** to four-vector components with upper indices. One also says that  $x^{\mu}$  **transforms contravariantly** and  $\partial_{\mu} \phi$

**transforms covariantly.** Vector components with upper indices are thus called **contravariant components** and such with lower ones **covariant components**.

One can always uniquely switch from contra- to covariant components and vice versa, by the **index-dragging rule**

$$x_{\mu} = \eta_{\mu\nu} x^{\nu}; \Rightarrow x^{\nu} = \eta^{\mu\nu} x_{\mu}. \quad (1.1.62)$$

We only have to show that  $x_{\mu}$  indeed transforms like covariant components should:

$$x'_{\mu} = \eta_{\mu\nu} x'^{\nu} = \eta_{\mu\nu} \Lambda^{\nu}_{\sigma} x^{\sigma} = \eta_{\mu\nu} \eta^{\sigma\rho} \Lambda^{\nu}_{\sigma} x_{\rho} \stackrel{(1.1.57)}{=} (\Lambda^{-1})^{\rho}_{\mu} x_{\rho}. \quad (1.1.63)$$

Comparing this with (1.1.58) shows that  $x_{\rho}$  indeed transforms contragrediently to  $x^{\rho}$  as it should.

For completeness we also give the rotation-free Lorentz transformation for a reference frame  $\Sigma'$  which moves with an arbitrary velocity  $\vec{v}$  ( $|\vec{v}| < 1$ ) against a frame  $\Sigma$ . This is easily found by looking at (1.1.24) and setting  $\vec{e}_x \rightarrow \vec{n} = \vec{v}/|\vec{v}|$ :

$$t' = \gamma_v(t - \vec{v} \cdot \vec{x}), \quad \vec{x}' = \vec{x} + (\gamma_v - 1)\vec{n}(\vec{n} \cdot \vec{x}) - \gamma_v \vec{v}t. \quad (1.1.64)$$

The corresponding Lorentz-transformation matrix can be written in the form

$$\hat{\Lambda}(\vec{v}) = \begin{pmatrix} \gamma_v & -\gamma_v \vec{v}^T \\ -\gamma_v \vec{v} & \mathbb{1} + (\gamma_v - 1)\vec{n} \otimes \vec{n} \end{pmatrix}. \quad (1.1.65)$$

We finally note that for time-like vectors, e.g., the four-momentum of a particle  $p^{\mu}$ , we can always find a reference frame, where  $\vec{p}' = 0$ . We only have to use the Lorentz boost (1.1.65) with a velocity  $\vec{v} = \vec{p}/\sqrt{p_{\mu}p^{\mu}}$ . Further also the sign of the temporal component  $p^0$  is invariant under Lorentz transformations. One should keep in mind that the composition of two Lorentz boosts with non-collinear velocities do not result in another Lorentz boost but a Lorentz boost followed by a rotation!

## 1.2 The Phase-Space Distribution Function

In this Section we shall discuss fundamental ideas of statistical physics. Historically statistical physics goes back to the work of physicists like Bernoulli, Gibbs, Boltzmann, Maxwell, Einstein, and others. The idea is to describe the behavior of **macroscopic systems** by a **coarse grained description** of the behavior of its **microscopic constituents**, obeying fundamental dynamical laws as given by classical mechanics and the fundamental interactions (which at the time were just gravitation and the electromagnetic interaction). Here we shall concentrate on the most simple case of **dilute gases**.

Of course, the 18<sup>th</sup>- and 19<sup>th</sup>-century physicists had no idea about the revolutionary development of quantum theory in the early 20<sup>th</sup> century, and they described the motion of the fundamental constituents of matter (atoms or molecules) as small bodies (or point particles) moving under the influence of forces among the particles (leading to collisions) according to the laws of Newtonian mechanics. From this point of view statistical physics becomes just necessary due to the vast number of microscopic constituents making up the gas in a container under everyday conditions of pressure and temperature. It is simply impossible to write down the **microscopic state of the system** which consists of  $3N$  position coordinates and  $3N$  momentum coordinates, where  $N \sim N_A \sim 10^{24}$  is the number of particles making up the gas and follow their trajectories in this  $6N$ -dimensional **phase space** with time, let alone to solve the coupled differential equations of motion, taking into account the interactions of particles among themselves and with the wall of the container, etc. It is not even sensible to do so, since

what we want to know in practice is a more **continuum like description**, i.e., the time evolution of the **density, flow velocity, pressure and temperature of the gas** as function of time and position in the container in the spirit of a **continuum-mechanical description**. As we shall discuss later in these lectures, there are different levels of description, depending on the way we average over microscopic degrees of freedom to obtain the macroscopic bulk quantities mentioned above. One level is **kinetic theory**, which soon shall lead us to the **Boltzmann transport equation**.

Here, the general idea is that for the description of the macroscopic properties of the gas we may define a “grid” in phase space consisting of volume elements  $d^3\vec{x}$  and  $d^3\vec{p}$  in position and momentum space, which are “infinitesimally small” on a macroscopic scale, i.e., on a scale over which the bulk properties like the density of gas particles and the momentum-distribution of particles, taken on average over such a cell, containing “many” particles, change.

Already at this point the question arises, what the microscopic measure for phase space might be, and here we have to refer to **quantum theory**. Assuming that the particles in a (rarefied) gas mostly move freely and interact only occasionally via short-ranged interactions, we use the model of a many-body system of non-interacting particles in a typical macroscopically small but microscopically large phase-space volume  $d^6\xi = d^3\vec{x}d^3\vec{p}$ . Quantum theoretically we describe such a many-body system in terms of the Fock space of relativistic local quantum field theory. The appropriate single-particle basis are **momentum eigenstates** of particles in a cubic volume of length  $L$  (such that  $L^3 = d^3\vec{x}$ ). To have a proper notion of a momentum operator, defined as the generator of spatial translations (as part of the Poincaré group) we should impose periodic boundary conditions. This leads to the discrete momentum eigenvalues

$$\vec{p} = \frac{2\pi\hbar}{L}\vec{n}, \quad \vec{n} \in \mathbb{Z}^3. \quad (1.2.1)$$

Thus the phase-space volume element  $d^6\xi_j = d^3\vec{x}d^3\vec{p} = L^3d^3\vec{p}_j$  represents

$$dG_j = \frac{g d^6\xi_j}{(2\pi\hbar)^3} \quad (1.2.2)$$

single-particle states, where  $g$  is the degeneracy according to spin, i.e.,  $2s + 1$  for massive and 1 or 2 for massless particles with spin  $s = 0$  or  $s \geq 1/2$ , respectively.

Then we describe the system by the **phase-space distribution**,  $f(t, \vec{x}, \vec{p})$ , which is defined such that

$$dN = d^3\vec{x}d^3\vec{p} \frac{g}{(2\pi\hbar)^3} f(t, \vec{x}, \vec{p}) \quad (1.2.3)$$

is the number of particles contained in the phase-space volume  $d^6\xi = d^3\vec{x}d^3\vec{p}$  at the instant of time,  $t$ . All this is defined for an observer in an arbitrary inertial frame of reference.

It is clear that any observer in another frame of reference moving with velocity  $v$  against the original frame must count the same number of particles when he considers the same volume of the gas at the same instant of time. We shall show now, that the phase-space distribution function can be defined as a **Lorentz scalar** although this is not so obvious given that neither  $d^3\vec{x}$  nor  $d^3\vec{p}$  are invariant under Lorentz transformations.

However, the spatial momentum components  $\vec{p}$  together with  $p^0 = E = \sqrt{m^2 + \vec{p}^2}$  build a four-vector. Further the four-momentum volume element is a scalar since the volume element transforms with the Jacobian of the transformation,

$$d^4p' = d^4p \left| \det \frac{\partial(p'^0, p'^1, p'^2, p'^3)}{\partial(p^0, p^1, p^2, p^3)} \right| = d^4p |\det \hat{\Lambda}|, \quad (1.2.4)$$

but now, taking the determinant of (1.1.2), we find

$$\begin{aligned}\det \hat{\eta} &= \det(\hat{\Lambda} \hat{\eta} \hat{\Lambda}^T) = \det(\hat{\Lambda}) \det(\hat{\eta}) \det(\hat{\Lambda}^T) = \det(\hat{\eta}) [\det(\hat{\Lambda})]^2 \\ &\Rightarrow [\det(\hat{\Lambda})]^2 = 1 \Rightarrow |\det \hat{\Lambda}| = 1.\end{aligned}\quad (1.2.5)$$

Thus (1.2.4) gives

$$d^4 p' = d^4 p. \quad (1.2.6)$$

Now in our classical particle picture all four-momenta of particles are “on shell”, i.e., the temporal component is  $p^0 = \sqrt{m^2 + \vec{p}^2} = E_p$ . This means that we can use a distribution function  $f(t, \vec{x}, \vec{p}) = f(t, \vec{x}, p^0, \vec{p})$  and set  $p^0 = E_p$  using a  $\delta$  distribution. This leads to the substitution

$$d^3 \vec{p} \rightarrow d^4 p \delta(p^0 - E_p). \quad (1.2.7)$$

The  $\delta$  distribution can be rewritten in manifestly covariant form, because

$$\Theta(p^0) \delta(p \cdot p - m^2) = \frac{1}{2E_p} \delta(p^0 - E_p). \quad (1.2.8)$$

Now, if we restrict the Lorentz transformations to **orthochronous transformations**, i.e., such transformations for which the **time ordering of events** is preserved, which means that  $\Lambda^0_0 \geq 1$ , implying that the sign of the time components of time-like vectors like  $p$ , i.e., such vectors for which  $p \cdot p > 0$ , stays preserved. Also  $p \cdot p$  is a Lorentz invariant and thus we realize that for particles on their mass shell

$$\frac{d^3 \vec{p}}{E_p} = \frac{d^3 \vec{p}'}{E_{p'}} \quad (1.2.9)$$

is a Lorentz invariant.

Further to count the particles with momentum  $\vec{p}$  in a little momentum volume  $d^3 \vec{p}$  we can use a Lorentz boost with velocity  $\vec{v} = \vec{p}/E$  to transform the particle’s three-momentum to 0 at a time  $t$ , the **comoving reference frame**. Four-vector components with respect to this frame of reference will be denoted with a star in the following. Then we can look at any volume element  $d^3 \vec{x}^*$  at a fixed time  $dt^*$  and count the particles in this volume element. This number is

$$dN = d^3 \vec{x}^* d^3 \vec{p}^* \frac{g}{(2\pi \hbar)^3} f^*(x^*, \vec{p}^* = 0). \quad (1.2.10)$$

Now we want to express the **proper volume element**  $d^3 \vec{x}^*$  in terms of the reference frame of the original observer, where the particle has momentum  $\vec{p}$ . To this end we again use the invariance of the four-volume element:

$$d^4 x^* = dt^* d^3 \vec{x}^* = d^4 x = dt d^3 \vec{x}. \quad (1.2.11)$$

Now we have  $dt^* = d\tau = dt/\gamma_v = dt m/E_p$  so that

$$d^3 \vec{x}^* = \frac{dt}{dt^*} d^3 \vec{x} = \frac{E_p}{m} d^3 \vec{x}. \quad (1.2.12)$$

Since, according to (1.2.9)

$$\frac{d^3 \vec{p}^*}{E_{p^*}} = \frac{d^3 \vec{p}^*}{m} = \frac{d^3 \vec{p}}{E_p} \Rightarrow d^3 \vec{p}^* = \frac{m}{E_p} d^3 \vec{p} \stackrel{(1.2.12, 1.2.13)}{\Rightarrow} d^3 \vec{x}^* d^3 \vec{p}^* = d^3 \vec{x} d^3 \vec{p}, \quad (1.2.13)$$



we find from (1.2.10)

$$dN = d^3\vec{x}d^3\vec{p} \frac{g}{(2\pi\hbar)^3} f(x, \vec{p}) = d^3\vec{x}^*d^3\vec{p}^* \frac{g}{(2\pi\hbar)^3} f(x, \vec{p}) \Rightarrow f(x, \vec{p}) \stackrel{(1.2.10)}{=} f^*(x^*, \vec{p}^* = 0), \quad (1.2.14)$$

and thus that indeed  $f(x, \vec{p}) \equiv f(x, p)$  is a Lorentz-invariant function for on-shell particles,  $p^0 = E_p$ . As an example we look at the **canonical equilibrium-distribution function**. As we shall derive in the next Sections from the Boltzmann equation, the equilibrium state is characterized by a temperature and a chemical potential, both of which are **defined as Lorentz-scalar quantities**. In the case of equilibrium of a gas, there exists a preferred reference frame, where the fluid cells are at rest. As we shall see soon, in this case the equilibrium distribution is given by the **Maxwell-Jüttner distribution**

$$f^*(x^*, p^*) = \exp\left(-\frac{E_{p^*} - \mu}{T}\right). \quad (1.2.15)$$

Now we like to transform this to a reference frame, where the fluid cell is moving with velocity  $\vec{v}$ . Using the appropriate Lorentz boost we find that

$$p^{*0} = E_{p^*} = \gamma_v(E_p - \vec{v} \cdot \vec{p}) = u \cdot p \quad \text{with} \quad u = \gamma_v \left( \frac{1}{\vec{v}} \right). \quad (1.2.16)$$

Thus in an arbitrary frame we have

$$f(x, p) = \exp\left(-\frac{p \cdot u - \mu}{T}\right). \quad (1.2.17)$$

We can now express **macroscopic quantities** in a covariant way by momentum integrations over the phase-space distribution, taking appropriate weights. Using the invariance of  $d^3\vec{p}/E_p$ , cf. (1.2.9), we can define the **four-vector current**

$$J^\mu(x) = \int_{\mathbb{R}^3} d^3\vec{p} \frac{g}{(2\pi\hbar)^3} \frac{p^\mu}{E_p} f(x, p). \quad (1.2.18)$$

Since always  $p^0 = E_p$  the temporal component gives the **particle-number density** and the spatial components the **particle-number current**

$$\begin{aligned} J^0(x) &= \int_{\mathbb{R}^3} d^3\vec{p} \frac{g}{(2\pi\hbar)^3} f(x, p), \\ \vec{J}(x) &= \int_{\mathbb{R}^3} d^3\vec{p} \frac{g}{(2\pi\hbar)^3} \frac{\vec{p}}{E_p} f(x, p) = \int_{\mathbb{R}^3} d^3\vec{p} \frac{g}{(2\pi\hbar)^3} \vec{v} f(x, p). \end{aligned} \quad (1.2.19)$$

Obviously  $N^\mu(x)$  is a timelike four-vector, because from  $|\vec{v}| = |\vec{p}|/E_p < 1$  follows

$$J_\mu J^\mu > 0. \quad (1.2.20)$$

Thus we can define the **flow-velocity field** by the average velocity of the fluid cell in the sense of this particle-number flow, the so-called **Eckart definition of flow**:

$$\vec{v}_{\text{Eck}}(x) = \frac{\vec{J}(x)}{J^0(x)}. \quad (1.2.21)$$

This is, of course not a Lorentz-covariant description of the fluid-flow field. As for the single-particle velocity we can, however, introduce a fluid-flow four-vector field,

$$u_{\text{Eck}}^\mu = \gamma_{v_{\text{Eck}}} \begin{pmatrix} 1 \\ \vec{v}_{\text{Eck}} \end{pmatrix} = \frac{J^\mu}{\sqrt{J_\mu J^\mu}} =: \frac{J^\mu}{n}. \quad (1.2.22)$$

The latter form shows the Lorentz-covariance of this four-velocity field explicitly. Obviously  $n = \sqrt{J_\mu J^\mu}$  is the **particle-number density in the local restframe of the fluid** in the sense of the Eckart framework, i.e., in the frame of reference, where  $\vec{v}_{\text{Eck}} = 0$ . From (1.2.22) we have the relation between the particle-number four-current and the Eckart-four-flow velocity,

$$J^\mu(x) = n(x) u_{\text{Eck}}^\mu(x). \quad (1.2.23)$$

If there are no particle-annihilation or production processes involved in the dynamics of the many-body system, the total number of particles is conserved, i.e., for any finite volume  $V$  (at rest in the considered reference frame) the change of number of particles inside this volume can only originate from the flow of particles through its boundary surface,  $\partial V$ . As usual, we orient the surface-normal vectors  $d^2\vec{F}$  always to point outside of the volume  $V$ . Then we have

$$\frac{dN_V}{dt} = \frac{d}{dt} \int_V d^3\vec{x} J^0(x) = \int_V d^3\vec{x} \partial_t J^0(x) = - \int_{\partial V} d^2\vec{F} \cdot \vec{J}(x). \quad (1.2.24)$$

The right-hand side of the equation obviously gives the number of particles per unit time running through the surface of the volume (counted negative if flowing outwards and positive if flowing inwards). Now we can use Gauss's integral theorem on the right-hand side to write the surface integral as a volume integral and combine it with the left-hand side,

$$\int_V d^3\vec{x} [\partial_t J^0(x) + \vec{\nabla} \cdot \vec{J}] = \int_V d^3\vec{x} \partial_\mu J^\mu = 0. \quad (1.2.25)$$

Since this relation holds for any resting volume, we can conclude the **local covariant form** of particle-number conservation, i.e.,

$$\partial_\mu J^\mu = \partial_t J^0 + \vec{\nabla} \cdot \vec{J} = 0, \quad (1.2.26)$$

which is nothing else than the **continuity equation of the current**, describing the conservation of the corresponding **total charge**. Here this "charge" is simply the particle number.

We also emphasize that the naively calculated total number of particles,

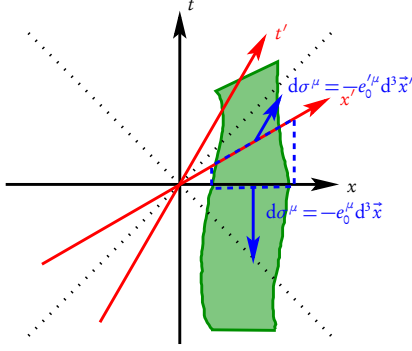
$$N = \int_{\mathbb{R}^3} d^3\vec{x} J^0(\vec{x}) \quad (1.2.27)$$

is a **Lorentz-scalar quantity** if (1.2.26) holds. This is proven as follows. First, from (1.2.24) by choosing the boundary  $\partial V$  at infinity the right-hand side vanishes, and for two inertial observers using space-time coordinates  $x$  and  $x'$  respectively, we have

$$\frac{d}{dt} N(t) = 0, \quad \frac{d}{dt'} N'(t') = 0, \quad (1.2.28)$$

i.e., for both observers the total particle number, defined by (1.2.27) is conserved.

## 1.2 · The Phase-Space Distribution Function



To show that the total particle number is also a scalar, i.e., that  $N' = N$ , we use the *four-dimensional* Gauss's integral theorem applied to the four-volume  $V^{(4)}$  depicted in the figure, where the shaded region denotes the evolving fluid, where the particle density and current is non-zero. Then one has

$$\begin{aligned} 0 &= \int_{V^{(4)}} d^4x \partial_\mu J^\mu \\ &= \int_{\partial V^{(4)}} d^3\sigma_\mu J^\mu \\ &= \int_{t'=0} d^3\vec{x}' J'^0(t=0, \vec{x}') - \int_{t=0} d^3\vec{x} J^0(t=0, \vec{x}), \end{aligned} \quad (1.2.29)$$

which indeed implies that

$$N' - N = 0 \Rightarrow N' = N. \quad (1.2.30)$$

Another important quantity is the **energy-momentum-stress tensor**,

$$T^{\mu\nu}(x) = T^{\nu\mu} = \int_{\mathbb{R}^3} d^3\vec{p} \frac{g}{(2\pi\hbar)^3} \frac{p^\mu p^\nu}{E_p} f(x, p). \quad (1.2.31)$$

For  $\mu = \nu = 0$  we get

$$T^{00}(x) = \int_{\mathbb{R}^3} d^3\vec{p} \frac{g}{(2\pi\hbar)^3} E_p f(x, p), \quad (1.2.32)$$

which is the **energy density** of the fluid. For  $\mu = k \in \{1, 2, 3\}$  and  $\nu = 0$  we find

$$T^{k0} = T^{0k} := S^k = \int_{\mathbb{R}^3} d^3\vec{p} \frac{g}{(2\pi\hbar)^3} p^k f(x, p), \quad (1.2.33)$$

the **momentum density** of the fluid.

With the very same arguments as for the particle-number current we can conclude that for a closed system, for which energy and momentum are conserved, the energy-momentum tensor obeys the continuity equation

$$\partial_\mu T^{\mu\nu} = \partial_t T^{0\nu} + \partial_k T^{k\nu} = 0. \quad (1.2.34)$$

Integrating this equation over a resting volume  $V$  and using Gauss's theorem for the second term yields

$$\frac{dP_V^\nu}{dt} = - \int_{\partial V} d^2F^k T^{k\nu} = - \int_{\partial V} d^2F^k T^{\nu k}. \quad (1.2.35)$$

Here, on the right-hand side we use the Einstein convention for usual three-dimensional vector-tensor products. That is why in this case we sum over two superscript indices. We can as well write

$$\frac{dP_V^\nu}{dt} = + \int_{\partial V} d^2F^k T^{\nu}_k. \quad (1.2.36)$$

For  $\nu = 0$  we get

$$\frac{dE_V}{dt} = - \int_{\partial V} d\vec{F} \cdot \vec{S}. \quad (1.2.37)$$

This shows that the loss of energy contained in the fixed volume  $V$  is due to momentum flow through the surface. This also shows that the momentum-density vector  $\vec{S}$  is at the same time the energy-current density. For  $\nu = j \in \{1, 2, 3\}$  we find that

$$\frac{d\vec{p}_V}{dt} = \int_{\partial V} d^2\vec{F} \cdot \hat{\sigma}, \quad (1.2.38)$$

where the three-dimensional Euclidean tensor

$$\sigma^{kj} = \sigma^{jk} = T^j_k = -T^{jk} \quad (1.2.39)$$

is the **stress tensor**.

### 1.3 The Relativistic Boltzmann Equation

The **Boltzmann equation** attempts at finding an (approximate) evolution equation for the phase-space distribution function  $f(t, \vec{x}, \vec{p})$ . As we have discussed in the previous Section, this is a Lorentz-invariant phase-space distribution function, where  $dN(t) = g/(2\pi\hbar)^3 f(t, \vec{x}, \vec{p}) d^3\vec{x} d^3\vec{p}$  is the number of particles in a phase-space volume  $d^6\xi = d^3x d^3p$ , as counted by an observer in an arbitrary inertial frame of reference. This number changes with time because of **collisions** among the particles. In the following we shall assume **short-range interactions** between the particles, i.e., within the resolution of a **macroscopically determined position**,  $\vec{x}$ , the collision occurs locally in space, i.e., at the position given by  $\vec{x}$ .

The change of this particle number  $dN(t)$  with time is now given by

$$\frac{(2\pi\hbar)^3}{g} dN(t+dt) = f(t+dt, \vec{x} + dt d\vec{x}/dt, \vec{p} + dt d\vec{p}/dt) d^6\xi(t+dt). \quad (1.3.1)$$

The change of  $f$  is thus

$$df = dt \left( \frac{\partial}{\partial t} + \frac{d\vec{x}}{dt} \cdot \frac{\partial}{\partial \vec{x}} + \frac{d\vec{p}}{dt} \cdot \frac{\partial}{\partial \vec{p}} \right) f. \quad (1.3.2)$$

Now for a particle moving under influence of an external force, e.g., due to an external electromagnetic field, we have

$$\frac{d\vec{x}}{dt} = \frac{1}{\gamma_v} \frac{d\vec{x}}{d\tau} = \frac{\vec{p}}{m} \frac{m}{E} = \frac{\vec{p}}{E} \quad (1.3.3)$$

and

$$\frac{d\vec{p}}{dt} = \vec{F}(t, \vec{x}, \vec{p}). \quad (1.3.4)$$

Thus (1.3.2) can be written as

$$df = dt \left( \frac{\partial}{\partial t} + \frac{\vec{p}}{E} \cdot \frac{\partial}{\partial \vec{x}} + \vec{F} \cdot \frac{\partial}{\partial \vec{p}} \right) f. \quad (1.3.5)$$

The phase-space volume element at  $t+dt$  is given in terms of the one at time  $t$  with help of the Jacobian

$$d^6\xi(t+dt) = d^6\xi(t) \det \left( \frac{\partial(\vec{x} + dt \vec{p}/E, \vec{p} + dt \vec{F})}{\partial(\vec{x}, \vec{p})} \right) = d^6\xi(t) \left( 1 + dt \frac{\partial}{\partial \vec{p}} \cdot \vec{F} \right) + \mathcal{O}(dt^2). \quad (1.3.6)$$

Thus we have

$$dN(t+dt) - dN(t) = d^6\xi(t) dt \left[ \frac{\partial f}{\partial t} + \frac{\vec{p}}{E} \cdot \frac{\partial \vec{f}}{\partial \vec{x}} + \frac{\partial(\vec{F}f)}{\partial \vec{p}} \right] \quad (1.3.7)$$

Since the left-hand side is an invariant, also the right-hand side should be one. Thus, for completeness, we show that we can write it in a manifestly covariant way. Indeed, for a particle with momentum  $\vec{p}$  the proper-time increment along its trajectory reads  $d\tau = dt/\gamma_v$  or  $dt = \gamma_v d\tau = (E/m)d\tau$ . Thus we have

$$dt \left[ \frac{\partial f}{\partial t} + \frac{\vec{p}}{E} \cdot \frac{\partial \vec{f}}{\partial \vec{x}} \right] = d\tau \frac{p^\mu}{m} \frac{\partial f}{\partial x^\mu}, \quad (1.3.8)$$

which is a manifestly covariant expression. Further we have

$$dt \frac{\partial(\vec{F}f)}{\partial \vec{p}} = \frac{E}{m} d\tau \frac{\partial(\vec{F}f)}{\partial \vec{p}}. \quad (1.3.9)$$

We rewrite this expression in terms of the covariant Lorentz force (1.1.38). Because of (1.1.40)

$$p^\mu K_\mu = 0 \Rightarrow p^0 K^0 = \vec{p} \cdot \vec{K} \Rightarrow K^0 = \frac{\vec{p}}{p^0} \cdot \vec{K}. \quad (1.3.10)$$

Because of the on-shell condition,  $p^0 = E_p = \sqrt{m^2 + \vec{p}^2}$ , we can think of  $p^0$  first as an independent variable in  $K^\mu(x, p)$  and then, using  $\partial E_p / \partial \vec{p} = \vec{p}/E$ , write

$$\begin{aligned} \frac{E}{m} \frac{\partial(\vec{F}f)}{\partial \vec{p}} &= \frac{E}{m} \left( \frac{\vec{p}}{E} \frac{\partial}{\partial p^0} + \frac{\partial}{\partial \vec{p}} \right) \cdot \left( \frac{m \vec{K} f}{p^0} \right) \\ &= \frac{\partial}{\partial p^0} \left( \frac{\vec{p} \cdot \vec{K} f}{p^0} \right) + \frac{\partial}{\partial \vec{p}} \cdot (\vec{K} f) \\ &\stackrel{(1.3.10)}{=} \frac{\partial}{\partial p^0} (K^0 f) + \frac{\partial}{\partial \vec{p}} \cdot (\vec{K} f) = \frac{\partial}{\partial p^\mu} (K^\mu f), \end{aligned} \quad (1.3.11)$$

setting  $p^0 = E_p$  after all derivatives are taken. Thus, substitution of (1.3.8) and (1.3.9) in (1.3.7) finally leads to

$$dN(t+dt) - dN(t) = dt \left[ \frac{\partial f}{\partial t} + \frac{\vec{p}}{E} \cdot \frac{\partial \vec{f}}{\partial \vec{x}} + \frac{\partial(\vec{F}f)}{\partial \vec{p}} \right] = d\tau \left[ \frac{p^\mu}{m} \frac{\partial f}{\partial x^\mu} + \frac{\partial(K^\mu f)}{\partial p^\mu} \right]. \quad (1.3.12)$$

Thus we have

$$\frac{d}{dt} dN = \frac{d^6\xi(t)}{E} \left[ p^\mu \frac{\partial f}{\partial x^\mu} + m \frac{\partial(K^\mu f)}{\partial p^\mu} \right]. \quad (1.3.13)$$

If there were no collisions between the particles this expression would vanish, because the particle number in an invariant phase-space element  $d^6\xi$  would stay constant.

Now, with collisions at position  $\vec{x}$ , within a time interval  $dt$  there is a certain amount  $dN^-$  of particles with momentum  $\vec{p}$  scattered to other momentum values  $\vec{p}'$  (“**loss term**”) but as well a number  $dN^+$

of particles with momentum  $\vec{p}'$  might get scattered into particles with momentum  $\vec{p}$  (“**gain term**”). This means that

$$\frac{d}{dt}dN = \frac{d}{dt}(dN^+ - dN^-). \quad (1.3.14)$$

To evaluate the number of particles scattering in and out of the phase-space element  $d^6\xi$  we have to make some simplifying assumptions, known as **Boltzmann’s Stoßzahlansatz**<sup>3</sup>. It consists of the following postulates

1. The gas is assumed to be so dilute that only the **elastic scattering** of two particles are relevant, because the collision of three particles within one fluid cell is extremely unlikely.
2. The two incoming particles with momenta  $\vec{p}_1$  and  $\vec{p}_2$  are **uncorrelated**, as well as the two outgoing particles with momenta  $\vec{p}'_1$  and  $\vec{p}'_2$  are uncorrelated, i.e., the two-particle phase-space density is given by the product of the corresponding one-particle densities,

$$f^{(2)}(t, \vec{x}, \vec{p}_1, \vec{p}_2) = f(t, \vec{x}, \vec{p}_1)f(t, \vec{x}, \vec{p}_2) := f_1f_2. \quad (1.3.15)$$

This is the **assumption of molecular chaos**.

3. The phase-space distribution function varies slowly over a time interval which is on the one hand much smaller than the time between two collisions (“mean free time” of a particle) but on the other hand much larger than the duration of a collision (“collision time”), i.e., the time where the influence of the interaction is effective for the particle’s motion. This separation of scales implies that the interaction should be **short ranged**. The assumption is, e.g., violated when long-range forces like the **Coulomb interaction** for charged particles in a plasma are taken into account.

Now we have to care about the question, how to describe the scattering. In our classical picture, the two incoming particles with momenta  $\vec{p}_1$  and  $\vec{p}_2$  start out very far from each other so that the interaction between the particles can be neglected until they come close to each other within the range of the interaction. Then they exchange energy and momentum and flow away from each other. When they become much farther away from each other, as compared to the interaction range, they again fly away independently of each other with new momenta  $\vec{p}'_1$  and  $\vec{p}'_2$ . The energies are always given by the onshell-condition, and energy and momentum are conserved after the interaction is ineffective again after the collision. This **energy-momentum conservation** we can write in terms of the four-momenta as

$$p_1 + p_2 = p'_1 + p'_2. \quad (1.3.16)$$

To characterize the collision, we want to define a quantity, which tells us how the two particles are scattered, i.e., how the momentum changes occur, the so-called **invariant scattering-cross section**, i.e., a quantity which is invariant under Lorentz transformations. The number of elastic collisions of two particles with four-momenta  $p_1$  and  $p_2$  to be scattered to momenta  $p'_1$  and  $p'_2$  during a short time interval  $dt$  should be proportional to the densities of the incoming particles, i.e.,  $g^2/(2\pi\hbar)^6 d^3\vec{p}_1 d^3\vec{p}_2 f_1 f_2$ , where  $f_j = f(x, \vec{p}_j)$ , the volume element  $d^3\vec{x}$ , and the time interval  $dt$ , i.e.,  $d^4x$ . In order to get an invariant characterization of this number, we define it in terms of an invariant **transition rate per unit volume**,

$$dN_{\text{coll}}(p'_1, p'_2 \leftarrow p_1, p_2) = d^4x \frac{d^3\vec{p}_1}{E_1} \frac{d^3\vec{p}_2}{E_2} \frac{d^3\vec{p}'_1}{E'_1} \frac{d^3\vec{p}'_2}{E'_2} \frac{g^2}{(2\pi\hbar)^6} f_1 f_2 W(p'_1, p'_2 \leftarrow p_1, p_2). \quad (1.3.17)$$

<sup>3</sup>Stoßzahlansatz is German for collision-number assumption.

Thus the collision term reads

$$\frac{g}{(2\pi\hbar)^3} \frac{d}{dt} (dN^+ - dN^-) = \frac{g^2}{(2\pi\hbar)^6} \int \frac{d^3\vec{p}_2}{E_2} \frac{d^3\vec{p}'_1}{E'_1} \frac{d^3\vec{p}'_2}{E'_2} \left[ f'_1 f'_2 W(p, p_2 \leftarrow p'_1, p'_2) - f_1 f_2 W(p'_1, p'_2 \leftarrow p_1, p_2) \right]. \quad (1.3.18)$$

Next we shall show, how this rate is related to the **invariant cross section**. First of all we note that due to (1.3.16) we must have  $W(p'_1, p'_2 \leftarrow p_1, p_2) \propto \delta^{(4)}(p'_1 + p'_2 - p_1 - p_2)$ . To characterize the scattering kinematics further in an invariant way, we introduce the **Mandelstam variables**<sup>4</sup>

$$s = (p_1 + p_2)^2 = (p'_1 + p'_2)^2, \quad t = (p_1 - p'_1)^2 = (p_2 - p'_2)^2, \quad u = (p_1 - p'_2)^2 = (p'_1 - p_2)^2. \quad (1.3.19)$$

From the on-shell conditions we conclude that these three quantities are not independent of each other. Multiplying out the Minkowski products gives

$$\begin{aligned} s &= 2m^2 + 2p_1 \cdot p_2, & t &= 2m^2 - 2p_1 \cdot p'_1, & u &= 2m^2 - 2p_1 \cdot p'_2 \\ &\Rightarrow s + t + u = 6m^2 + 2p_1 \cdot (p_2 - p'_1 - p'_2) = 6m^2 - 2p_1^2 = 4m^2. \end{aligned} \quad (1.3.20)$$

Thus we have two independent invariants to characterize the collision, e.g.,  $s$  and  $t$ .

Obviously  $s = 2m^2 + 2(E_1 E_2 - \vec{p}_1 \cdot \vec{p}_2) > 0$ , because  $\vec{p}_1 \cdot \vec{p}_2 \leq P_1 P_2$  and  $E_1 E_2 > P_1 P_2$ . Thus  $p_1 + p_2$  is a time like four-vector, and we can always use the Lorentz boost with velocity  $\vec{v} = (\vec{p}_1 + \vec{p}_2)/(E_1 + E_2)$  to make  $\vec{p}_1^{(\text{cm})} + \vec{p}_2^{(\text{cm})} = 0$ , defining the **center-momentum frame**. This implies that

$$s = (E_1^{(\text{cm})} + E_2^{(\text{cm})})^2, \quad (1.3.21)$$

i.e.  $\sqrt{s}$  is the **total energy of the incoming particles as measured in the center-momentum frame**. Because  $P_1^{(\text{cm})} = P_2^{(\text{cm})}$  we also have

$$\begin{aligned} E_1^{(\text{cm})} &= E_2^{(\text{cm})} = E_1'^{(\text{cm})} = E_2'^{(\text{cm})} = \frac{\sqrt{s}}{2} \\ &\Rightarrow P_1^{(\text{cm})} = P_2^{(\text{cm})} = P_1'^{(\text{cm})} = P_2'^{(\text{cm})} = \sqrt{\frac{s}{4} - m^2} = \frac{\sqrt{s - 4m^2}}{2}. \end{aligned} \quad (1.3.22)$$

Here we have used that due to momentum conservation also in the final state we have  $\vec{p}'_1^{(\text{cm})} = -\vec{p}'_2^{(\text{cm})}$ . Traditionally, however, in high-energy physics the **scattering cross section** is defined in the so-called **laboratory frame** since in the early days of particle accelerators the usual experiments were **fixed-target experiments**, where one particle in the initial state is at rest. Thus we define the laboratory frame by

$$\vec{p}_2^{(\text{lab})} = 0. \quad (1.3.23)$$

Since  $p_2^2 = m^2 > 0$  we can always use the Lorentz boost with velocity  $\vec{v} = \vec{p}_1/E_1$  to transform from an arbitrary frame to this laboratory frame. In terms of the laboratory momenta we have

$$s = (p_1 + p_2)^2 = 2m^2 + 2p_1 \cdot p_2 = 2m^2 + 2mE_1^{(\text{lab})}, \quad (1.3.24)$$

<sup>4</sup>For a four-vector  $p$ , we use the abbreviation  $p^2 = p \cdot p = \eta_{\mu\nu} p^\mu p^\nu$ .

i.e., the energy of the incoming particle in the laboratory frame is given by

$$E_1^{(\text{lab})} = \frac{s - 2m^2}{2m} \quad (1.3.25)$$

and the lab momentum

$$P_1^{(\text{lab})} = |\vec{p}_1^{(\text{lab})}| = \sqrt{[E_1^{(\text{lab})}]^2 - m^2} = \frac{\sqrt{s(s - 4m^2)}}{2m} = \frac{\sqrt{s}}{m} P_1^{(\text{cm})}. \quad (1.3.26)$$

In the center-momentum frame the collision kinematics is further completely determined by the **scattering angle**, which is defined by

$$\cos \theta^{(\text{cm})} = \frac{\vec{p}_1^{(\text{cm})} \cdot \vec{p}'_1^{(\text{cm})}}{P_1^{(\text{cm})} P'_1^{(\text{cm})}}. \quad (1.3.27)$$

It can be written in an invariant way by using the Mandelstam- $t$  variable:

$$t = (p_1^{(\text{cm})} - p'_1^{(\text{cm})})^2 = 2m^2 - 2[(E_1^{(\text{cm})})^2 - (P_1^{(\text{cm})})^2 \cos^2 \theta^{(\text{cm})}]. \quad (1.3.28)$$

Using (1.3.22) we find

$$\cos \theta^{(\text{cm})} = \frac{2t}{s - 4m^2} - 1. \quad (1.3.29)$$

The **invariant differential cross section** is now defined in the laboratory frame as the number of particles per unit time scattered into a solid-angle range  $d\Omega^{(\text{lab})}$  (where the solid angle is defined in terms of spherical coordinates in the laboratory frame with the polar axis defined by the direction of the momentum of the incoming particles,  $\vec{p}_1$ , as  $d\Omega^{(\text{lab})} = d\vartheta d\varphi \sin \vartheta$ ) per **incoming flux of the projectiles** and **per particle density in the target region**. According to (1.3.17) this is given by

$$d\sigma = \frac{W(p'_1, p'_2 \leftarrow p_1, p_2) d^4x \frac{d^3\vec{p}_1}{E_1} \frac{d^3\vec{p}_2}{m} \frac{d^3\vec{p}'_1}{E'_1} \frac{d^3\vec{p}'_2}{E'_2} \frac{g^2}{(2\pi\hbar)^6} f_1 f_2}{d^4x \frac{g^2}{(2\pi\hbar)^6} d^3\vec{p}_1 v_{\text{rel}} f_1 d^3\vec{p}_2 f_2}, \quad (1.3.30)$$

where all momenta are to be taken as momenta in the **laboratory frame**, where the incoming particle 2 is at rest, i.e.,  $E_2 = m$  and  $\vec{p}_2 = 0$ . The relative velocity is defined as the three-velocity of particle 1 in this laboratory frame,

$$v_{\text{rel}} := \left| \frac{\vec{p}_1^{(\text{lab})}}{E_1^{(\text{lab})}} \right|. \quad (1.3.31)$$

Now to show that  $d\sigma$  is indeed Lorentz invariant, we write (1.3.30) and (1.3.31) in a covariant form by using

$$p_1^{(\text{lab})} \cdot p_2^{(\text{lab})} = E_1^{(\text{lab})} m \Rightarrow v_{\text{rel}} = \frac{\sqrt{(p_1 \cdot p_2)^2 - m^4}}{E_1^{(\text{lab})} m} =: \frac{I}{p_1 \cdot p_2}. \quad (1.3.32)$$

This shows that the so defined differential cross section is indeed invariant,

$$d\sigma = \frac{d^3\vec{p}'_1}{E'_1} \frac{d^3\vec{p}'_2}{E'_2} \frac{W(p'_1, p'_2 \leftarrow p_1, p_2)}{I}. \quad (1.3.33)$$



The dimension of  $d\sigma$  in our natural units is  $\text{GeV}^{-2}$ . In usual SI units it is an area<sup>5</sup>. Now we take into account the energy-momentum-conserving  $\delta$  distribution and write

$$W(p'_1, p'_2 \leftarrow p_1, p_2) = s \frac{d}{d\Omega^{(\text{cm})}} \sigma(s, t) \delta^{(4)}(p_1 + p_2 - p'_1 - p'_2). \quad (1.3.34)$$

Then we have

$$\sigma_{\text{tot}} = \frac{1}{2} \int_{\mathbb{R}^3} \frac{d^3 \vec{p}'_1}{E'_1} \int_{\mathbb{R}^3} \frac{d^3 \vec{p}'_2}{E'_2} \frac{s}{I} \frac{d}{d\Omega^{(\text{cm})}} \sigma(s, t) \delta^{(4)}(p_1 + p_2 - p'_1 - p'_2). \quad (1.3.35)$$

The factor 1/2 is due to the **indistinguishability of the particles**, i.e., we cannot distinguish the here considered scattering event from a situation, where the final momenta are interchanged. To integrate out the  $\delta$  distributions we evaluate the integral in the center-momentum frame. The integration over  $\vec{p}'_1^{(\text{cm})}$  is immediately done, ensuring four-momentum conservation, which we tacitly assume in the following, i.e., we have

$$\sigma_{\text{tot}} = \frac{1}{2} \int_{\mathbb{R}^3} \frac{d^3 \vec{p}'_2^{(\text{cm})}}{E_2^{(\text{cm})}} \frac{s}{I E_1^{(\text{cm})}} \frac{d}{d\Omega^{(\text{cm})}} \sigma(s, t) \delta \left[ \sqrt{s} - 2 \sqrt{(P_2^{(\text{cm})})^2 + m^2} \right], \quad (1.3.36)$$

where we have used that  $E_1^{(\text{cm})} + E_2^{(\text{cm})} = \sqrt{s}$  and  $E_1^{(\text{cm})} = E_2^{(\text{cm})}$ . Now we introduce spherical coordinates and use

$$\delta \left[ \sqrt{s} - 2 \sqrt{(P_2^{(\text{cm})})^2 + m^2} \right] = \frac{E_2^{(\text{cm})}}{2 P_2^{(\text{cm})}} \delta \left( P_2^{(\text{cm})} - \frac{\sqrt{s - 4m^2}}{2} \right), \quad (1.3.37)$$

$$I \stackrel{(1.3.32)}{=} \sqrt{(p_1 \cdot p_2)^2 - m^4} = \frac{\sqrt{s}}{2} \sqrt{s - 4m^2}. \quad (1.3.38)$$

which indeed leads to

$$\sigma_{\text{tot}} = \frac{1}{2} \int_{S_2} d\Omega^{\text{cm}} \frac{d}{d\Omega^{(\text{cm})}} \sigma(s, t). \quad (1.3.39)$$

Here we note that we have defined the total cross section as the integral of the differential cross section over the total solid angle (unit sphere  $S_2$ ), although due to the indistinguishability of the particles we should have integrated only over half the solid angle.

Finally we note that from (1.3.34) we find the **detailed-balance relation**

$$W(p'_1, p'_2 \leftarrow p_1, p_2) = W(p_1, p_2 \leftarrow p'_1, p'_2), \quad (1.3.40)$$

because obviously a simple interchange of the initial momenta with the final momenta doesn't change the Mandelstam variables,  $s$  and  $t$ .

With this convention we finally can write down the **Boltzmann Equation**. We start from (1.3.13) and (1.3.14) using (1.3.17) and (1.3.40) for both the gain and the loss rates of particles in the phase-space volume  $d^6 \xi^6$

$$p^\mu \frac{\partial f_1}{\partial x^\mu} + m \frac{\partial (K^\mu f_1)}{\partial p^\mu} = \frac{1}{2} \frac{g}{(2\pi \hbar)^3} \int_{\mathbb{R}^3} \frac{d^3 \vec{p}_2}{E_2} \int_{\mathbb{R}^3} \frac{d^3 \vec{p}'_1}{E'_1} \int_{\mathbb{R}^3} \frac{d^3 \vec{p}'_2}{E'_2} \times W(p'_1, p'_2 \leftarrow p, p_2) (f'_1 f'_2 - f_1 f_2), \quad (1.3.41)$$

<sup>5</sup>One can simply convert from natural to SI units by multiplying with the appropriate power of  $\hbar c \simeq 0.197 \text{ GeV fm}$ . For the cross section one multiplies by  $(\hbar c)^2$  to obtain the cross section in  $\text{fm}^2 = 10^{-30} \text{m}^2$ . Another typical unit for cross sections are milli-barns,  $1 \text{mb} = 10^{-31} \text{m}^2 = 0.1 \text{fm}^2$ .

<sup>6</sup>Note that we have cancelled one factor  $g/(2\pi \hbar)^3$  on both the left- and the right-hand side of the equation.

where the factor  $1/2$  is again due to the indistinguishability of the particles. This is the Boltzmann Equation in manifestly covariant notation. Using (1.3.12) and  $dt/d\tau = E/m$  we find the more conventional form

$$\begin{aligned} \frac{\partial f}{\partial t} + \frac{\vec{p}_1}{E_1} \cdot \frac{\partial f}{\partial \vec{x}} + \frac{\partial(\vec{F}f)}{\partial \vec{p}_1} &= \frac{1}{2} \frac{g}{(2\pi\hbar)^3} \\ &\times \int_{\mathbb{R}^3} d^3\vec{p}_2 \int_{\mathbb{R}^3} d^3\vec{p}'_1 \int_{\mathbb{R}^3} d^3\vec{p}'_2 w(p'_1, p'_2 \leftarrow p_1, p_2) (f'_1 f'_2 - f_1 f_2), \end{aligned} \quad (1.3.42)$$

where we have introduced the **non-covariant transition rate**,

$$w(p'_1, p'_2 \leftarrow p, p_2) = \frac{W(p'_1, p'_2 \leftarrow p, p_2)}{E_1 E_2 E'_1 E'_2}. \quad (1.3.43)$$

The integration over  $\vec{p}'_1$  and  $\vec{p}'_2$  can be simplified in the same way as in our above evaluation of  $\sigma_{\text{tot}}$ , using (1.3.34) in (1.3.41) which leads to

$$p^\mu \frac{\partial f_1}{\partial x^\mu} + m \frac{\partial(K^\mu f_1)}{\partial p^\mu} = \frac{1}{2} \frac{g}{(2\pi\hbar)^3} \int_{\mathbb{R}^3} \frac{d^3\vec{p}_2}{E_2} \int_{S_2} d\Omega^{(\text{cm})} I \frac{d\sigma}{d\Omega^{(\text{cm})}} (f'_1 f'_2 - f_1 f_2). \quad (1.3.44)$$

For the non-covariant form we have to introduce the **Møller velocity**,

$$v_M = \frac{I}{EE_2}, \quad (1.3.45)$$

leading to

$$\frac{\partial f_1}{\partial t} + \frac{\vec{p}}{E} \cdot \frac{\partial f_1}{\partial \vec{x}} + \frac{\partial(\vec{F}f_1)}{\partial \vec{p}} = \frac{1}{2} \frac{g}{(2\pi\hbar)^3} \int_{\mathbb{R}^3} d^3\vec{p}_2 \int_{S_2} d\Omega^{(\text{cm})} v_M \frac{d\sigma}{d\Omega^{(\text{cm})}} (f'_1 f'_2 - f_1 f_2). \quad (1.3.46)$$

It is very important to remember that  $v_M$ , often called  $v_{\text{rel}}$  in the literature, is only the naive relative velocity, if the incoming momenta are collinear as, e.g., in the center-momentum frame. For the general case we find

$$v_M = \frac{I_1}{EE_2} = \frac{\sqrt{(p_1 \cdot p_2)^2 - m^4}}{E_1 E_2}. \quad (1.3.47)$$

By straightforward but lengthy algebra (*Exercise!*) one can show that this can be rewritten as

$$v_M = \sqrt{(\vec{v} - \vec{v}_2)^2 - (\vec{v} \times \vec{v}_2)^2}, \quad (1.3.48)$$

i.e., indeed the Møller velocity is identical with the “naive relative velocity” if and only if  $\vec{v} \times \vec{v}_2 = 0$ , i.e., for collinear velocities.

For completeness we note the relation between the above used invariant quantity  $W$  to the usual invariant matrix elements  $\mathcal{M}$  in high-energy quantum field theory. Using the convention in [O<sup>+</sup>14], the unpolarized differential cross section (1.3.33) is written in the form

$$d\sigma = \frac{|\overline{\mathcal{M}_{fi}}|^2}{4I} (2\pi)^4 \delta^{(4)}(p + p_2 - p'_1 - p'_2) \frac{d^3\vec{p}'_1}{(2\pi)^3 2E'_1} \frac{d^3\vec{p}'_2}{(2\pi)^3 2E'_2}, \quad (1.3.49)$$

where  $\overline{|\mathcal{M}_{fi}|^2}$  is the squared matrix element, *averaged* over the spin states of the incoming two particles and *summed* over the spin states of the (in our case of elastic scattering also two) outgoing particles of the collision. Also the natural-unit convention  $\hbar = 1$  is employed.

Comparing (1.3.49) with (1.3.33) yields the relation

$$W(p'_1, p'_2 \leftarrow p_1, p_2) = \frac{\overline{|\mathcal{M}_{fi}|^2}}{16(2\pi)^6} (2\pi)^4 \delta^{(4)}(p + p_2 - p'_1 - p'_2). \quad (1.3.50)$$

We close this section with a general note on the detailed-balance relation. For the more general case, where the collision term involves not only one particle species the strong relation (1.3.40) does not necessarily hold true. Nevertheless, for the following we only need the proof that we can simplify the collision term from the general form (1.3.18) to the right-hand side of (1.3.41). Now we shall show that this is justified by the generally valid **unitarity of the Scattering matrix**.

First we note the relation between the  $S$ - and the  $T$ -matrix elements

$$S_{fi} = \mathbb{1} - i(2\pi)^4 \delta^{(4)}(P_f - P_i) T_{fi} \quad (1.3.51)$$

with  $P_f$  and  $P_i$  denoting the sum over the four-momenta of the outgoing and incoming particles in the scattering process, respectively. The relation with the invariant matrix elements is given by

$$T_{fi} = \frac{\mathcal{M}_{fi}}{\prod_{k \in \{f, i\}} [(2\pi)^3 2E_k]^{1/2}}, \quad (1.3.52)$$

where the product runs over all momenta in the initial and final states.

To simplify the argument, we assume that we first describe the particles in a finite quantization four-volume, leading to discrete three-momenta according to the here appropriate periodic boundary conditions, as employed already above when counting the number of single-particle microstates within a macroscopic phase-space cell. Then the energy-momentum conserving  $\delta$  distribution in (1.3.51) become a usual Kronecker  $\delta$ , and there is no problem in exploiting the unitarity relations for the  $S$ -Matrix

$$\hat{S}^\dagger \hat{S} = \hat{S} \hat{S}^\dagger = \mathbb{1}. \quad (1.3.53)$$

With the Kronecker  $\delta$  lumped into the  $T$ -Matrix, setting  $\hat{S} = \mathbb{1} - i\hat{T}$  then gives

$$\hat{S}^\dagger \hat{S} = (\mathbb{1} + i\hat{T}^\dagger)(\mathbb{1} - i\hat{T}) = \mathbb{1} + i(\hat{T}^\dagger - \hat{T}) + \hat{T}^\dagger \hat{T} = \mathbb{1} \Rightarrow \hat{T}^\dagger \hat{T} = -i(\hat{T}^\dagger - \hat{T}) \quad (1.3.54)$$

and

$$\hat{S} \hat{S}^\dagger = (\mathbb{1} - i\hat{T})(\mathbb{1} + i\hat{T}^\dagger) = \mathbb{1} + i(\hat{T}^\dagger - \hat{T}) + \hat{T} \hat{T}^\dagger = \mathbb{1} \Rightarrow \hat{T} \hat{T}^\dagger = -i(\hat{T}^\dagger - \hat{T}). \quad (1.3.55)$$

Taken Eqs. (1.3.54) and (1.3.55) together yields

$$\hat{T}^\dagger \hat{T} = \hat{T} \hat{T}^\dagger. \quad (1.3.56)$$

Written in terms of matrix elements this relation reads

$$\sum_f T_{fi}^* T_{fi} = \sum_f T_{if} T_{if}^*. \quad (1.3.57)$$

Making the quantization volume again the entire Minkowski space-time  $\mathbb{R}^4$ , we find for the here interesting two-particle final-state integral

$$\int d^3\vec{p}'_1 \int d^3\vec{p}'_2 \delta^{(4)}(P_f - P_i) |T_{fi}|^2 = \int d^3\vec{p}'_1 \int d^3\vec{p}'_2 \delta^{(4)}(P_f - P_i) |T_{if}|^2. \quad (1.3.58)$$

In terms of the invariant function  $W(p'_1, p'_2 \leftarrow p_1, p_2)$ , as it occurs in the loss term in (1.3.18), we have

$$\int \frac{d^3\vec{p}'_1}{E'_1} \int \frac{d^3\vec{p}'_2}{E'_2} W(p'_1, p'_2 \leftarrow p_1, p_2) = \int \frac{d^3\vec{p}'_1}{E'_1} \int \frac{d^3\vec{p}'_2}{E'_2} W(p_1, p_2 \leftarrow p'_1, p'_2), \quad (1.3.59)$$

and this is sufficient to write (1.3.18) in the form (1.3.42), which is all what's needed in the following. This very general derivation shows that this “**weak form**” of **detailed balance** holds true for any scattering processes, i.e., it is not restricted to the here considered elastic  $2 \rightarrow 2$  scattering processes with only one species of particles involved. Also it is not relying on the parity and time-reversal symmetries of the interactions considered. As we shall see below, the weak detailed-balance principle is sufficient to derive the Boltzmann  $H$ -theorem, i.e., the monotonous growing of the entropy with time, which thus holds for time-reversal symmetric as well as unsymmetric interactions<sup>7</sup>.

## 1.4 The Master Equation and Conservation Laws

In the following we shall derive a master equation for the currents of various conserved quantities, leading to macroscopic descriptions of matter in the sense of fluid dynamics.

We start from the Boltzmann equation in its manifestly covariant form (1.3.41). Multiplying it by an arbitrary phase-space function  $\psi(x, p)$  and integrating over  $\vec{p}$  with the invariant measure  $d^3\vec{p}/E$  yields<sup>8</sup>

$$\int_{\mathbb{R}^3} \frac{d^3\vec{p}}{E} \psi(x, p) \left[ p^\mu \frac{\partial f}{\partial x^\mu} + m \frac{\partial (K^\mu f)}{\partial p^\mu} \right] = \text{Coll}[\psi] \quad (1.4.1)$$

with the **collision functional**

$$\begin{aligned} \text{Coll}[\psi] = & \frac{1}{2} \frac{g}{(2\pi\hbar)^3} \int_{\mathbb{R}^3} \frac{d^3\vec{p}}{E} \int_{\mathbb{R}^3} \frac{d^3\vec{p}_2}{E_2} \int_{\mathbb{R}^3} \frac{d^3\vec{p}'_1}{E'_1} \int_{\mathbb{R}^3} \frac{d^3\vec{p}'_2}{E'_2} \\ & \times \psi(x, p) W(p'_1, p'_2 \leftarrow p, p_2) (f'_1 f'_2 - f f_2). \end{aligned} \quad (1.4.2)$$

Because of the detailed-balance relation (1.3.40) we can interchange the four-momentum pairs  $(p_1, p_2)$  and  $(p'_1, p'_2)$  in the first term of this equation and write

$$\text{Coll}[\psi] = \frac{1}{2} \frac{g}{(2\pi\hbar)^3} \int_{\mathbb{R}^3} \frac{d^3\vec{p}}{E} \int_{\mathbb{R}^3} \frac{d^3\vec{p}_2}{E_2} \int_{\mathbb{R}^3} \frac{d^3\vec{p}'_1}{E'_1} \int_{\mathbb{R}^3} \frac{d^3\vec{p}'_2}{E'_2} (\psi'_1 - \psi) W(p'_1, p'_2 \leftarrow p, p_2) f f_2, \quad (1.4.3)$$

where we use the same abbreviations  $\psi = \psi(x, p)$ ,  $\psi_2 = \psi(x, p_2)$ , etc. as for the distribution functions  $f$ . Finally, because of the indistinguishability of the particles we can also interchange the four-momentum pairs  $(p, p'_1)$  and  $(p_2, p'_2)$ . Adding this expression and dividing again by two, we finally get

<sup>7</sup>The weak interaction breaks both parity and time-reversal as well as  $CP$  invariance.

<sup>8</sup>Remember that the momenta are always considered to be on the mass shell, i.e.,  $p^0 = E_p = \sqrt{m^2 + \vec{p}^2}$ !

$$\text{Coll}[\psi] = \frac{1}{4} \frac{g}{(2\pi\hbar)^3} \int_{\mathbb{R}^3} \frac{d^3\vec{p}}{E} \int_{\mathbb{R}^3} \frac{d^3\vec{p}_2}{E_2} \int_{\mathbb{R}^3} \frac{d^3\vec{p}'_1}{E'_1} \int_{\mathbb{R}^3} \frac{d^3\vec{p}'_2}{E'_2} (\psi'_1 + \psi'_2 - \psi - \psi_2) W(p'_1, p'_2 \leftarrow p, p_2) f f_2. \quad (1.4.4)$$

Now we consider the left-hand side of (1.4.1). The first term we can rewrite as

$$\int_{\mathbb{R}^3} \frac{d^3\vec{p}}{E} \psi(x, p) p^\mu \frac{\partial f}{\partial x^\mu} = \frac{\partial}{\partial x^\mu} \int_{\mathbb{R}^3} \frac{d^3\vec{p}}{E} f \psi p^\mu - \int_{\mathbb{R}^3} \frac{d^3\vec{p}}{E} f p^\mu \frac{\partial \psi}{\partial x^\mu}. \quad (1.4.5)$$

To simplify the second term we have to remember the definition of the derivatives with respect to  $p^\mu$ . They have to be read in the sense to first take the derivatives of the covariantly written expression with  $p^0$  considered as independent and only *after the differentiation* set  $p^0 = E$ . Thus the meaning of the second expression under the integral on the left-hand side of (1.4.1) is

$$\begin{aligned} \frac{m}{E} \psi \frac{\partial(K^\mu f)}{\partial p^\mu} &\stackrel{(1.3.11)}{=} \psi \frac{\partial(f\vec{F})}{\partial \vec{p}} = \frac{\partial}{\partial \vec{p}} \cdot (\psi f \vec{F}) - f \vec{F} \cdot \frac{\partial}{\partial \vec{p}} \psi(x, E, \vec{p}) \\ &= \frac{\partial}{\partial \vec{p}} \cdot (\psi f \vec{F}) - f \vec{F} \cdot \left( \frac{\vec{p}}{E} \frac{\partial}{\partial p^0} - \frac{\partial}{\partial \vec{p}} \right) \psi(x, p) \\ &\stackrel{(1.3.10)}{=} \frac{\partial}{\partial \vec{p}} \cdot (\psi f \vec{F}) - \frac{m}{E} K^\mu \frac{\partial \psi}{\partial p^\mu}. \end{aligned} \quad (1.4.6)$$

Integrating over  $\vec{p}$  and using Gauss's integral theorem on the first term to rewrite the integral as an integral over a surface in  $\vec{p}$  space at infinity, which makes this contribution vanish, gives

$$\int_{\mathbb{R}^3} \frac{d^3\vec{p}}{E} m \psi \frac{\partial(K^\mu f)}{\partial p^\mu} = - \int_{\mathbb{R}^3} \frac{d^3\vec{p}}{E} m K^\mu \frac{\partial \psi}{\partial p^\mu}. \quad (1.4.7)$$

Now using (1.4.5), (1.4.7), and (1.4.4) in (1.4.1) gives the **general master equation**

$$\begin{aligned} &\frac{\partial}{\partial x^\mu} \int_{\mathbb{R}^3} \frac{d^3\vec{p}}{E} p^\mu f \psi - \int_{\mathbb{R}^3} \frac{d^3\vec{p}}{E} f \left( p^\mu \frac{\partial \psi}{\partial x^\mu} + m K^\mu \frac{\partial \psi}{\partial p^\mu} \right) \\ &= \frac{1}{4} \frac{g}{(2\pi\hbar)^3} \int_{\mathbb{R}^3} \frac{d^3\vec{p}}{E} \int_{\mathbb{R}^3} \frac{d^3\vec{p}_2}{E_2} \int_{\mathbb{R}^3} \frac{d^3\vec{p}'_1}{E'_1} \int_{\mathbb{R}^3} \frac{d^3\vec{p}'_2}{E'_2} (\psi'_1 + \psi'_2 - \psi - \psi_2) \\ &\quad \times W(p'_1, p'_2 \leftarrow p, p_2) f f_2. \end{aligned} \quad (1.4.8)$$

This expression becomes particularly simple, if the collision functional on the right-hand side vanishes, which is the case if  $\psi(x, p)$  fulfills the functional equation

$$\psi + \psi_2 = \psi'_1 + \psi'_2, \quad (1.4.9)$$

where we can assume the energy-momentum-conservation constraints in the elastic scattering process, because of the corresponding  $\delta$  distributions contained in  $W$  (cf. 1.3.34). A function which fulfills (1.4.9) under the constraint of energy-momentum conservation is called a **summational invariant**.

We can find the general form of the invariant by introducing the constraints via Lagrange parameters for the energy-momentum constraints, which can be dependent on  $x$ , and then taking the variations with respect to  $\vec{p}$ ,  $\vec{p}_2$ ,  $\vec{p}'_1$ , and  $\vec{p}'_2$

$$\frac{\vec{p}}{E} \frac{\partial \psi}{\partial p^0} + \frac{\partial \psi}{\partial \vec{p}} = \lambda^0 \frac{\vec{p}}{E} - \vec{\lambda}. \quad (1.4.10)$$

Setting  $\lambda^\mu = B^\mu(x)$ , this is solved by

$$\psi(x, p) = A(x) + B_\mu(x)p^\mu, \quad (1.4.11)$$

where the on-shell constraint  $p^0 = E$  is tacitly assumed again.

Particularly, choosing  $A(x) = 1$ ,  $B_\mu = 0$ , i.e.,  $\psi(x, p) = 1$  we find from (1.4.8) the **conservation of particle number** in form of the continuity equation for the particle-flow four-current density follows:

$$\frac{\partial}{\partial x^\mu} \int_{\mathbb{R}^3} \frac{d^3 \vec{p}}{E} p^\mu f \stackrel{(1.2.18)}{=} \frac{\partial J^\mu}{\partial x^\mu} = 0. \quad (1.4.12)$$

Setting  $A(x) = 0$  and  $B_\mu(x) = \delta_\mu^\alpha$ , i.e.,  $\psi(x, p) = p^\alpha$  we find from (1.4.8)

$$\frac{\partial}{\partial x^\mu} \int_{\mathbb{R}^3} \frac{d^3 \vec{p}}{E} p^\alpha p^\mu f = \int_{\mathbb{R}^3} \frac{d^3 \vec{p}}{E} f m K^\alpha = \int_{\mathbb{R}^3} d^3 \vec{p} f F^\alpha \quad \text{with} \quad F^\alpha = \frac{m}{E} K^\alpha. \quad (1.4.13)$$

The spatial parts of the right-hand side of this equation have the meaning of a **force density**.

For  $\alpha = 0$  (1.4.14) has the meaning of a local energy-balance equation. Due to the constraint (1.1.40) we have

$$p_\mu K^\mu = EK^0 - \vec{p} \cdot \vec{K} = 0 \Rightarrow K^0 = \frac{\vec{p}}{E} \cdot \vec{K} = \vec{v} \cdot \vec{K} \Rightarrow F^0 = \frac{m}{E} K^0 = \vec{v} \cdot \vec{F}. \quad (1.4.14)$$

Thus indeed for  $\alpha = 0$  the right-hand side of (1.4.13) becomes the meaning of the density of mechanical power due to the work of the external forces on a fluid element around  $\vec{x}$ .

The spatial components  $\alpha = a \in \{1, 2, 3\}$  of (1.4.13) take the form of a local equation of motion for the fluid element at  $x$  subject to external forces.

In the same way one can derive the angular-momentum-torque relation by setting  $\psi(x, p) = x^\alpha p^\beta - x^\beta p^\alpha$  (*Exercise!*).

## 1.5 The entropy and the $H$ -Theorem

To define entropy, even in the classical limit, the cleanest way is to consider quantum theory. According to (1.2.2) in the phase-space volume  $d^6 \xi_j$  there are

$$G_j = \frac{g d^6 \xi_j}{(2\pi \hbar)^3} \quad (1.5.1)$$

single-particle states, where  $g$  counts the intrinsic quantum numbers like spin, for which  $g = 2s + 1$  for massive particles or  $g_j = 2$  for the two helicity states of massless particles of any spin  $s_j \neq 0$ .

For a dilute gas, i.e., if the number  $N_j$  of particles contained in the phase-space element  $d^6 \xi_j$  is on average small compared to the number of quantum states,  $G_j$ , we can neglect the degeneracy according to quantum (Bose-Einstein or Fermi-Dirac) statistics. Then the statistical weight of the corresponding phase-space distribution is estimated as

$$d\Gamma_j = \frac{1}{N_j!} G_j^{N_j}. \quad (1.5.2)$$

The factorial in the denominator takes into account the **indistinguishability of particles**, i.e., it does not matter which individual particle of the  $N_j$  particles populates one of the  $G_j$  quantum states belonging to a macroscopic phase-space cell. So any distribution of the  $N_j$  particles which differs from

one such “macro state” only by permutation of all particles must be considered as the same  $N_j$ -particle state, and this is taken into account by the factorial in the denominator.

Following Boltzmann and Planck the **entropy** of the system for a given distribution of the  $N$  particles in phase space, given by the numbers  $N_j$  of particles in the phase-space cell  $d^6\xi_j$ , is defined as

$$S = \sum_j \ln d\Gamma_j = \sum_j [N_j \ln G_j - \ln(N_j!)] \simeq \sum_j [N_j \ln G_j - N_j(\ln N_j - 1)]. \quad (1.5.3)$$

In the last step we have used **Stirling’s approximation**,  $\ln(N!) \simeq_{N \rightarrow \infty} N(\ln N - 1)$ . Now we introduce the average number of particles per quantum state

$$n_j = \frac{N_j}{G_j}. \quad (1.5.4)$$

Then we can write (1.5.3) as

$$S = \sum_j N_j \ln \left( \frac{eG_j}{N_j} \right) = \sum_j n_j G_j \ln \left( \frac{e}{n_j} \right) = \sum_j \frac{g \Delta^6 \xi_j}{(2\pi \hbar)^3} n_j \ln \left( \frac{e}{n_j} \right). \quad (1.5.5)$$

Now in the limit of macroscopically small phase-space cells we can write

$$n_j = \frac{(2\pi \hbar)^3 N_j}{g \Delta^6 \xi_j} \simeq \frac{(2\pi \hbar)^3}{g} f(x, \vec{p}) = f(x, \vec{p}). \quad (1.5.6)$$

Thus the **entropy of a dilute gas** is given in terms of phase-space by the semiclassical expression

$$S(t) = - \frac{g}{(2\pi \hbar)^3} \int d^3 \vec{x} d^3 \vec{p} f(x, p) \{ \ln[f(x, p)] - 1 \}. \quad (1.5.7)$$

The above consideration shows that for a detailed foundation of classical statistical mechanics one needs quantum-theoretical arguments in order to give a proper definition of entropy in terms of the phase-space distribution function. The reason is that in classical physics there is no “natural measure” for the size of phase-space cells, i.e., one cannot unambiguously count the number of microscopic states leading to a given macroscopic distribution of particles in phase space. As we have shown above, this problem is quite simply solved by using basic concepts of quantum theory. The factor  $(2\pi \hbar)^3$  in the logarithm in (1.5.7) is important in order to make this argument dimensionless as it must be.

The relativistically covariant definition of the entropy uses the **entropy-density four-current**

$$S^\mu(x) = - \frac{g}{(2\pi \hbar)^3} \int_{\mathbb{R}^3} \frac{d^3 \vec{p}}{E} p^\mu f(x, p) \ln[f(x, p)] - 1 \}. \quad (1.5.8)$$

Boltzmann’s **H-theorem**<sup>9</sup> follows from the general master equation (1.4.8) by putting

$$\psi = \ln[f(x, p)] - 1. \quad (1.5.9)$$

Since here we have

$$\frac{\partial \psi}{\partial x^\mu} = - \frac{1}{f} \frac{\partial f}{\partial x^\mu}, \quad \frac{\partial \psi}{\partial p^\mu} = - \frac{1}{f} \frac{\partial f}{\partial p^\mu} \quad (1.5.10)$$

<sup>9</sup>Note that  $H$  stands for the capital greek letter Eta, Boltzmann’s nomenclature for entropy!

we find for the 2<sup>nd</sup> term on the left-hand side of (1.4.8), using the Boltzmann equation (1.3.41)

$$\int_{\mathbb{R}^3} \frac{d^3 \vec{p}}{E} f \left( p^\mu \frac{\partial \psi}{\partial x^\mu} + m K^\mu \frac{\partial \psi}{\partial p^\mu} \right) = - \int_{\mathbb{R}^3} \frac{d^3 \vec{p}}{E} \left( p^\mu \frac{\partial f}{\partial x^\mu} + m K^\mu \frac{\partial f}{\partial p^\mu} \right) = -\text{Coll}[1] \stackrel{(1.4.4)}{=} 0. \quad (1.5.11)$$

Thus the master equation for (1.5.9) gives

$$\frac{\partial S^\mu}{\partial x^\mu} = -\frac{1}{4} \frac{g}{(2\pi \hbar)^3} \int_{\mathbb{R}^3} \frac{d^3 \vec{p}}{E} \int_{\mathbb{R}^3} \frac{d^3 \vec{p}_2}{E_2} \int_{\mathbb{R}^3} \frac{d^3 \vec{p}'_1}{E'_1} \int_{\mathbb{R}^3} \frac{d^3 \vec{p}'_2}{E'_2} f f_2 \ln \left( \frac{f'_1 f'_2}{f f_2} \right) W(p'_1 p'_2 \leftarrow p, p_1). \quad (1.5.12)$$

On the other hand we can add half of the collision functional for  $\psi = 1$ , which vanishes identically due to (1.4.4), which gives

$$\begin{aligned} \frac{\partial S^\mu}{\partial x^\mu} := \zeta = & +\frac{1}{4} \frac{g}{(2\pi \hbar)^3} \int_{\mathbb{R}^3} \frac{d^3 \vec{p}}{E} \int_{\mathbb{R}^3} \frac{d^3 \vec{p}_2}{E_2} \int_{\mathbb{R}^3} \frac{d^3 \vec{p}'_1}{E'_1} \int_{\mathbb{R}^3} \frac{d^3 \vec{p}'_2}{E'_2} f f_2 \\ & \times \left[ \frac{f'_1 f'_2}{f f_2} - \ln \left( \frac{f'_1 f'_2}{f f_2} \right) - 1 \right] W(p'_1 p'_2 \leftarrow p, p_1). \end{aligned} \quad (1.5.13)$$

Now we discuss the function

$$f(z) = z - 1 - \ln z, \quad z > 0. \quad (1.5.14)$$

Obviously its derivative  $f'(z) = 1 - 1/z$  changes sign from negative to positive at  $z = 1$ . Thus the function is monotonously decreasing for  $z < 1$  and increasing for  $z > 1$ . At  $z = 1$  is thus a strict minimum with  $f(z) = 0$ . This means

$$f(z) \geq 0; \quad f(z) = 0 \Leftrightarrow z = 1. \quad (1.5.15)$$

From (1.5.13) we can thus conclude that the **entropy production is never negative**,  $\zeta \geq 0$ . Integrating over the entire spatial volume shows that the total entropy is **never decreasing with time**, and it is constant in time if and only if the square bracket of (1.5.13) vanishes.

## 1.6 Equilibrium Distributions

From this we can immediately derive the form of the **stationary solutions** of the Boltzmann equation, i.e., find the distribution functions which are time independent. From the demand, that the square bracket in (1.5.13) has to vanish, we can conclude that

$$\frac{f'_1 f'_2}{f f_2} = 1. \quad (1.6.1)$$

Taking the logarithm of this shows that

$$\phi = -\ln f \quad (1.6.2)$$

is a **summational invariant** in the sense of (1.4.9) and thus, according to (1.4.11) must be of the form

$$\phi = -\alpha(x) + \beta_\mu(x) p^\mu \Rightarrow f(x, p) = \exp[\alpha(x) - \beta(x) \cdot p]. \quad (1.6.3)$$

Since the particle-number-density four-current

$$J^\mu(x) = \frac{g}{(2\pi \hbar)^3} \int_{\mathbb{R}^3} \frac{d^3 \vec{p}}{E} p^\mu f(x, p) = n(x) u^\mu(x) \quad (1.6.4)$$



is a time-like four-vector we can introduce the Eckart definition of the four-velocity cf. (1.2.19-1.2.22). As shown in (1.2.22)  $n(x)$  has the physical meaning of the particle density in the local rest frame of the fluid. Since  $J^\mu$  is a four-vector and  $\beta^\mu$  is the only four vector in (1.6.3) we must have  $J^\mu \propto \beta^\mu$

$$\beta^\mu(x) = \beta(x)u^\mu(x) \quad \text{with} \quad u_\mu u^\mu \equiv 1. \quad (1.6.5)$$

In the local rest frame of the fluid, where  $u^\mu = (1, 0, 0, 0)$ , we must have  $\beta > 0$  in order to have a finite particle-number-density four-vector (1.6.4). It is also convenient to write

$$\alpha(x) = \beta(x)\mu(x), \quad (1.6.6)$$

so that the equilibrium distribution takes the **Boltzmann-Jüttner form**

$$f_{\text{eq}}(x, p) = \exp\left[-\beta(x)\left(u(x) \cdot p - \mu(x)\right)\right], \quad p^0 = E = \sqrt{m^2 + \vec{p}^2}. \quad (1.6.7)$$

Next we evaluate the energy-momentum tensor

$$T^{\mu\nu}(x) = \frac{g}{(2\pi\hbar)^3} \int_{\mathbb{R}^3} \frac{d^3\vec{p}}{E} p^\mu p^\nu f_{\text{eq}}(x, p). \quad (1.6.8)$$

Since this obviously is a symmetric tensor of 2<sup>nd</sup> rank, it only can be a linear combination of  $\eta^{\mu\nu}$  and  $u^\mu u^\nu$  since these are the only general tensors of this kind that can be built from the available building blocks of quantities available: The Minkowski metric is a Lorentz-invariant tensor, and the only other Lorentz-covariant quantity in our scheme that can be used to form a 2<sup>nd</sup>-rank tensor is  $u^\mu$ . We can build two independent invariants from this tensor by applying the projection matrices

$$P_{\parallel}^{\mu\nu} = u^\mu u^\nu, \quad P_{\perp}^{\mu\nu} = \Delta^{\mu\nu} = \eta^{\mu\nu} - u^\mu u^\nu \quad (1.6.9)$$

that allow to project on parallel and perpendicular components of the fluid-four velocity of arbitrary vectors in the sense of the Minkowski product. Thus we define

$$\epsilon = ne = T^{\mu\nu} u_\mu u_\nu, \quad P = -\frac{1}{3} \Delta_{\mu\nu} T^{\mu\nu}. \quad (1.6.10)$$

Then the energy-momentum tensor takes the form

$$T^{\mu\nu} = \epsilon u^\mu u^\nu - P \Delta^{\mu\nu} = (\epsilon + P)u^\mu u^\nu - P\eta^{\mu\nu}. \quad (1.6.11)$$

In the local rest frame it simplifies to<sup>10</sup>

$$T_*^{\mu\nu} = \text{diag}(\epsilon, P, P, P). \quad (1.6.12)$$

As discussed after Eq. (1.2.39) the purely spatial components of the energy-momentum tensor have the meaning of a **stress tensor**. In the local rest frame of the fluid according to (1.6.12) the stress tensor takes the form  $\sigma_*^{ab} = -P\delta^{ab}$ , i.e.,  $P$  is the **pressure of the gas**, which in equilibrium is locally isotropic in the local rest frame, as expected. According to (1.2.32) The purely temporal component of the energy-momentum tensor has the meaning of the energy density of the medium, and thus  $T_*^{00} = \epsilon = ne$  is the energy density of the medium in the local rest frame, i.e.,  $e$  is the **mean energy** of a single particle in

<sup>10</sup>In the following we denote vector and tensor components wrt. the local restframe with an asterisk subscript or superscript.

the gas, measured in the local rest frame, i.e., it is the **average thermal energy of one particle in the medium**.

Finally we evaluate the **entropy-current four vector**, cf. (1.5.8). Using (1.6.7) we find

$$S^\mu = \frac{g}{(2\pi\hbar)^3} \int_{\mathbb{R}^3} \frac{d^3\vec{p}}{E} f_{\text{eq}}(x, p) p^\mu [1 - \alpha(x) + \beta u_\nu(x) p^\nu] = [1 - \alpha(x)] J^\mu(x) + \beta T^{\mu\nu} u_\nu. \quad (1.6.13)$$

From this we find the entropy density in the local rest frame by contraction with  $u_\mu$ :

$$s = u_\mu S^\mu = (1 - \alpha)n + \beta\epsilon. \quad (1.6.14)$$

Now we can evaluate the various quantities defined in the local rest frame by setting  $u^\mu = (1, 0, 0, 0)$ . According to (1.6.4) particle density is given by

$$n = \frac{g}{(2\pi\hbar)^3} \exp(\alpha) \int_{\mathbb{R}^3} d^3\vec{p} \exp(-\beta\sqrt{m^2 + \vec{p}^2}). \quad (1.6.15)$$

Introducing polar coordinates  $\vec{p} = p(\cos\varphi \sin\vartheta, \sin\varphi \sin\vartheta, \cos\vartheta)$  we find

$$n = \frac{4\pi g}{(2\pi\hbar)^3} \exp(\alpha) \int_0^\infty dp p^2 \exp(-\beta\sqrt{m^2 + p^2}). \quad (1.6.16)$$

Substitution of  $p = m \sinh y$  gives

$$n = \frac{4\pi g m^3}{(2\pi\hbar)^3} \exp \alpha \int_0^\infty dy \cosh y \sinh^2 y \exp(-\beta m \cosh y). \quad (1.6.17)$$

We can evaluate this and the following integrals in terms of the modified Bessel functions, which we can define as

$$K_m(z) = \int_0^\infty dy \cosh(mz) \exp(-z \cosh y). \quad (1.6.18)$$

In the following we'll need some basic properties of these functions, which we will derive in Appendix 1.9.

Substitution of

$$\cosh y \sinh^2 y = \frac{\cosh(3y) - \cosh y}{4} \quad (1.6.19)$$

in (1.6.16) leads to

$$n = \frac{4\pi g m^2 \exp \alpha}{(2\pi\hbar)^3 \beta} K_2(m\beta), \quad (1.6.20)$$

where we have used the relation (1.9.2).

For the pressure we find, using (1.6.12) and (1.6.8) to get

$$P = \frac{1}{3} \sum_{a=1}^3 T_*^{aa} = \frac{g \exp \alpha}{3(2\pi\hbar)^3} \int_{\mathbb{R}^3} \frac{d^3\vec{p}}{E} \vec{p}^2 \exp(-\beta E). \quad (1.6.21)$$

In the same way as for the density we get (*Exercise!*)

$$P = \frac{4\pi g m^2 \exp \alpha}{(2\pi\hbar)^3 \beta^2} K_2(m\beta) = \frac{n}{\beta}. \quad (1.6.22)$$

Comparing with the well-known **ideal-gas equation of state**  $P = NT/V$  leads to

$$P = nT \Rightarrow T = \frac{1}{\beta}. \quad (1.6.23)$$

In this way we have identified the parameter  $\beta$  with the **temperature of the system**<sup>11</sup>.

To find the energy density, we only need to take the derivative of  $n$  with respect to  $\beta$  (at fixed  $\alpha$ ) leading to

$$\epsilon = - \left( \frac{\partial n}{\partial \beta} \right)_{\alpha=\text{const}} \stackrel{(1.9.4)}{=} \frac{4\pi g m^2 \exp \alpha}{(2\pi \hbar)^3 \beta^2} [m\beta K_1(m\beta) + 3K_2(m, \beta)]. \quad (1.6.24)$$

Using (1.6.23) in the entropy density (1.6.14) we can write it in the form

$$Ts = P + \epsilon - \mu n, \quad \mu = \alpha T. \quad (1.6.25)$$

We shall show now that  $\mu$  has the usual meaning of the **chemical potential**.

To that end we assume that  $\alpha$  and  $\beta$  are constants, i.e., not dependent on  $x$  in the rest frame of the fluid (**global thermal equilibrium**). Then we can write for the total energy, particle number, and entropy

$$U = V\epsilon, \quad S = Vs, \quad N = Vn, \quad (1.6.26)$$

respectively. Thus, multiplying (1.6.26) with  $V$  we find

$$U = TS - PV + \mu N = TS - NT + \mu N, \quad (1.6.27)$$

where in the last step we have used (1.6.14) in the form

$$pV = NT. \quad (1.6.28)$$

Now we can consider  $N$  as a function of  $V$ ,  $T$ , and  $\mu$ , using (1.6.15)

$$N(V, T, \mu) = Vn = \frac{gV}{(2\pi \hbar)^3} \exp(\mu/T) \int_{\mathbb{R}^3} d^3 \vec{p} \exp\left(-\frac{E}{T}\right). \quad (1.6.29)$$

From this we find

$$\partial_V N = \frac{N}{V}, \quad \partial_T N = \frac{U - \mu N}{T^2}, \quad \partial_\mu N = \frac{N}{T}, \quad (1.6.30)$$

which leads to

$$dN = \frac{N}{V} dV + \frac{U - \mu N}{T^2} dT + \frac{N}{T} d\mu \Rightarrow Nd\mu = TdN - \frac{NT}{V} dV - \frac{U - \mu N}{T} dT \quad (1.6.31)$$

or finally with (1.6.28)

$$Nd\mu = TdN - PdV - \frac{U - \mu N}{T} dT. \quad (1.6.32)$$

Now using (1.6.27) gives

$$dU = TdS - (S - N)dT + (\mu - T)dN + Nd\mu. \quad (1.6.33)$$

<sup>11</sup>Note that we use a system of units, where we measure temperatures in units of energy, i.e., we have set the Boltzmann constant  $k_B = 1$ .

Plugging in (1.6.32) for the last term and using (1.6.27) we find (*Exercise!*) the **First Law of Thermodynamics**:

$$dU = TdS - PdV + \mu dN. \quad (1.6.34)$$

This shows that any change of state in internal energy, keeping the system in equilibrium is due to the change of entropy ( $dQ = TdS$ : heat energy), the mechanical work of the gas against the pressure ( $dW = -PdV$ ), and due to a change in particle number ( $\mu dN$ ). Thus  $\mu$  is the amount of energy necessary to change the particle number within the volume by one particle, and this quantity is thus known as **chemical potential**.

For later use we note that from (1.6.33) and (1.6.27) we have

$$dU = TdS - PdV + \mu dN = d(TS - PV + \mu N) \Rightarrow SdT - VdP + Nd\mu = 0. \quad (1.6.35)$$

Dividing the latter Eq. by  $V$  we find the relation

$$sdT = dP - nd\mu. \quad (1.6.36)$$

Now we investigate the solutions of the Boltzmann equation (1.3.41). We know from (1.6.1) that the collision term vanishes identically for  $f = f_{\text{eq}}$ . Thus we have

$$p^\mu \frac{\partial f_{\text{eq}}}{\partial x^\mu} + mK^\mu \frac{\partial f_{\text{eq}}}{\partial p^\mu} = 0. \quad (1.6.37)$$

In the following we consider the case of vanishing external forces. For  $K^\mu = 0$  (1.6.37) simplifies to

$$p^\mu \frac{\partial f_{\text{eq}}}{\partial x^\mu} = 0. \quad (1.6.38)$$

Obviously the same equation holds for

$$\phi = \ln f = \phi \stackrel{(1.6.3)}{=} \alpha(x) + \beta_\mu(x)p^\mu. \quad (1.6.39)$$

This gives

$$p^\mu \frac{\partial \phi}{\partial x^\mu} = p^\mu \frac{\partial \alpha}{\partial x^\mu} + p^\mu p^\nu \frac{\partial \beta_\nu}{\partial x^\mu} = p^\mu \frac{\partial \alpha}{\partial x^\mu} + \frac{1}{2} p^\mu p^\nu \left( \frac{\partial \beta_\nu}{\partial x^\mu} + \frac{\partial \beta_\mu}{\partial x^\nu} \right) = 0. \quad (1.6.40)$$

In principle only the three spatial components of  $p^\mu$  are independent variables, because it is always understood that the on-shell condition  $p^0 = E = \sqrt{m^2 + \vec{p}^2}$  holds. Nevertheless, for  $\vec{p} = 0$  we have  $p^0 = m \neq 0$ , and thus (1.6.41) demands that for all  $\mu, \nu \in \{0, 1, 2, 3\}$

$$\frac{\partial \alpha}{\partial x^\mu} = 0, \quad \partial_\mu \beta_\nu + \partial_\nu \beta_\mu = 0, \quad (1.6.41)$$

where we have written  $\partial_\mu = \partial / \partial x^\mu$ . The first condition implies that

$$\alpha = \beta_\mu = \frac{\mu}{T} = \text{const.} \quad (1.6.42)$$

The second equation can be solved by differentiating it with respect to  $x^\rho$ , leading to

$$\partial_\rho \partial_\mu \beta_\nu + \partial_\rho \partial_\nu \beta_\mu = 0, \quad (1.6.43)$$

## 1.6 · Equilibrium Distributions

cyclically change the indices ( $\mu \rightarrow \nu, \nu \rightarrow \rho, \rho \rightarrow \mu$ ) to

$$\partial_\mu \partial_\nu \beta_\rho + \partial_\mu \partial_\rho \beta_\nu = 0 \quad (1.6.44)$$

and subtracting this equation from (1.6.44):

$$\partial_\nu (\partial_\rho \beta_\mu - \partial_\mu \beta_\rho) = 0. \quad (1.6.45)$$

This implies that

$$\partial_\rho \beta_\mu - \partial_\mu \beta_\rho = \frac{1}{2} \omega_{\mu\rho} = \text{const.} \quad (1.6.46)$$

together with the second equation in (1.6.41) this implies

$$\partial_\rho \beta_\mu = \omega_{\mu\rho} \Rightarrow \beta_\mu = \omega_{\mu\rho} x^\rho + a_\mu \quad \text{with} \quad a_\mu = \text{const.} \quad (1.6.47)$$

For a time-independent distribution, we must have  $\omega_{\mu 0} = -\omega_{0\mu} = 0$ . Then this distribution describes a fluid which is rigidly rotating with a constant angular velocity  $\omega^j = 1/2 \epsilon^{jkl} \omega^{kl}$ , and the most general **global-equilibrium distribution** reads

$$f_{\text{eq}}(x, p) = \frac{1}{(2\pi \hbar)^3} \exp(a \cdot p + \vec{\omega} \cdot \vec{J} + \alpha) \quad \text{with} \quad \vec{J} = \vec{x} \times \vec{p}. \quad (1.6.48)$$

Thus, the form of the global equilibrium function is completely governed by the conserved additive quantities from space-time symmetry, i.e., energy and momentum due to temporal and spatial translation invariance and angular momentum due to rotation invariance.

To further characterize the global equilibrium state we use

$$\beta^\mu = \frac{1}{T} u^\mu \quad (1.6.49)$$

in (1.6.41). The second equation leads to

$$T(\partial_\mu u_\nu + \partial_\nu u_\mu) = u_\mu \partial_\nu T + u_\nu \partial_\mu T. \quad (1.6.50)$$

Since  $u_\mu u^\mu = 1$  we have  $u_\mu \partial_\nu u^\mu = 0$  and thus multiplying (1.6.50) with  $u^\mu u^\nu$  gives

$$u^\mu \partial_\mu T = 0. \quad (1.6.51)$$

In the local rest frame  $u^\mu = (1, 0, 0, 0)$  this means that  $T$  is locally time-independent. In this sense  $u^\mu \partial_\mu$  can be interpreted as a **convective time derivative**.

Multiplying (1.6.50) only with  $u^\nu$  we find

$$u^\nu \partial_\nu u_\mu = \frac{1}{T} \partial_\mu T^\mu, \quad (1.6.52)$$

which tells us that in global equilibrium a **temperature gradient is compensated by an acceleration of the fluid elements**.

Finally, in the same way by setting

$$\alpha = \frac{\mu}{T} \quad (1.6.53)$$

in the first equation (1.6.41) and contracting with  $u^\mu$  we find that

$$u^\mu \partial_\mu \mu = 0, \quad (1.6.54)$$

i.e., the convective time derivative of the chemical potential also vanishes.

## 1.7 The relativistic perfect fluid

The limiting case of a **perfect fluid** applies, if the **mean-free path** of the fluid particles, i.e., the typical length a particle can move between two collisions, is very small compared to the typical length scales over which macroscopic properties change. Then we can assume that over the typical macroscopic scales the fluid is always in **local thermal equilibrium**.

As we have seen in the previous Section, the phase-space distribution of a fluid local equilibrium is characterized as a fluid motion, for which the entropy stays locally constant, which implied the general form of the equilibrium distribution as the Boltzmann-Jüttner distribution (1.6.7). It is determined by the fields  $T(x)$ ,  $\mu(x)$  and  $u^\mu(x)$ , i.e., temperature, chemical potential, and the fluid-flow vector. The latter obeys the constraint  $u_\mu u^\mu = 1$  and thus has only three independent components. This means we need also five independent **equations of motion** to specify the perfect-fluid flow completely. As we shall show now, these equations are given by the general conservation laws for particle number, energy, and momentum.

We find the hydrodynamic equations for a perfect fluid, analogous to the **Euler equations** for the non-relativistic case, by considering the general conservation laws for the particle number (1.4.12) and energy and momentum (1.4.13) for vanishing external forces,  $K^\mu = 0$ ,

$$\partial_\mu J^\mu = 0, \quad \partial_\mu T^{\mu\nu} = 0 \quad (1.7.1)$$

and plugging in the expressions for these quantities for a fluid in local thermal equilibrium (1.6.4) and (1.6.11) respectively

$$J^\mu = n u^\mu, \quad T^{\mu\nu} = (\epsilon + P) u^\mu u^\nu - P \eta^{\mu\nu}. \quad (1.7.2)$$

The unknown quantities to be determined in this formulation are the proper particle-number density  $n$ , the proper energy density  $\epsilon$ , pressure  $P$ , and the three spatial components of the fluid four-flow  $u^\mu$ . However, since in local thermal equilibrium we can express the number and energy densities as well as the pressure in terms of the temperature and chemical potential, we have an **equation of state**

$$P = P(\epsilon, n), \quad (1.7.3)$$

which closes the set of equations (1.7.1) for the five independent quantities  $n$ ,  $\epsilon$ , and  $\vec{u}$ .

For a better physical understanding of these equations of motion of the **perfect fluid** we decompose the space-time gradients in a covariant way into a temporal and a spacial derivative. Given the four-flow velocity field of the fluid we can write the time derivative in the **local restframe** of the fluid cell as

$$\partial_t^* = u^\mu \partial_\mu =: D \quad (1.7.4)$$

and decompose the four-gradient into this temporal and the spatial pieces wrt. the local restframe as

$$\partial_\mu = u_\mu D + \nabla_\mu, \quad (1.7.5)$$

where we can use the projection matrix (1.6.9) to define

$$\nabla_\mu = \Delta_{\mu\nu} \partial^\nu. \quad (1.7.6)$$

Applying this to the conservation of the particle number, i.e., the first Eq. (1.7.1) together with (1.7.2) one finds

$$\partial_\mu J^\mu = \partial(n u^\mu) = Dn + n \partial_\mu u^\mu = 0. \quad (1.7.7)$$

1.7 · The relativistic perfect fluid

On the other hand we have

$$\partial_\mu u^\mu = u_\mu D u^\mu + \nabla_\mu u^\mu = \nabla_\mu u^\mu, \quad (1.7.8)$$

because

$$u_\mu D u^\mu = \frac{1}{2} D(u_\mu u^\mu) = 0. \quad (1.7.9)$$

This leads to

$$Dn = -n \nabla_\mu u^\mu. \quad (1.7.10)$$

Next we take the “perpendicular part” of the energy-momentum conservation (2<sup>nd</sup> Eq. (1.7.1)) which together with the 2<sup>nd</sup> Eq. (1.7.2) leads to (*Exercise!*)

$$(\epsilon + P) D u_\mu = \nabla_\mu P. \quad (1.7.11)$$

Introducing the **enthalpy per particle**,

$$h = \frac{\epsilon + P}{n}, \quad (1.7.12)$$

we can write

$$nh D u_\mu = \nabla_\mu P. \quad (1.7.13)$$

For the spatial components,  $\mu \in \{1, 2, 3\}$  we can write

$$nh D \vec{u} = -\vec{\nabla} P + \vec{u} DP. \quad (1.7.14)$$

In the non-relativistic limit, due to  $u^0 = \gamma_v = 1/\sqrt{1 - \vec{v}^2}$  we have

$$\vec{v} = \vec{u} u^0 = \vec{v}[1 + \mathcal{O}(v^2)], \quad D = u^\mu \partial_\mu = (\partial_t + \vec{v} \cdot \vec{\nabla})[1 + \mathcal{O}(v^2)]. \quad (1.7.15)$$

Further we have

$$h = \frac{\epsilon + P}{n} = m[1 + \mathcal{O}(v^2)]. \quad (1.7.16)$$

Thus up to corrections of order  $\mathcal{O}(v^2)$  (1.7.15) reads

$$mn(\partial_t + \vec{v} \cdot \vec{\nabla})\vec{v} = -\vec{\nabla} P, \quad (1.7.17)$$

where on the right-hand side we can use the usual meaning of  $\vec{\nabla}$  as the spatial partial derivatives (again up to corrections of order  $\mathcal{O}(v^2)$ ). This is nothing else than **Euler’s equation of motion for an ideal fluid**. That we have in Eq. 1.7.14)  $nh$ , i.e., the enthalpy density and not the mass density as in the non-relativistic approximation (1.7.17), is due to the fact that in the relativistic equations of motion all forms of energy contribute to the **inertia** of the fluid element, which in here consists of the mass of the particles, their kinetic energy (which together contribute the density of the inner energy  $\epsilon = ne$ ), and the inner tension of the fluid, which is given by the pressure for the here considered ideal gas. On the right-hand side of (1.7.17) we have the force on the fluid element due to the tension (i.e., pressure) acting on this fluid element. In (1.7.14) there are additional relativistic correction terms which *must* occur in order to make the (1.7.13) consistent: Because of the constraint  $u_\mu u^\mu = 1$  we get  $u^\mu D u_\mu = 0$  and thus contracting the right-hand side of (1.7.13) with  $u^\mu$  must also give zero, and that’s precisely the case, because on the right-hand side covariant projection  $\nabla_\mu = \Delta_{\mu\nu} \partial^\nu$  of the four-gradient operator appears.

The projection of the 2<sup>nd</sup> Eq. (1.7.1) leads to

$$D\epsilon = -(\epsilon + P)\partial_\mu u^\mu = -nb\nabla_\mu u^\mu. \quad (1.7.18)$$

From (1.5.13) and the equilibrium condition (1.6.1) it follows that the perfect-fluid motion is always isentropic. This follows immediately from our derivation of the local-equilibrium distribution as the distribution of maximum entropy. Thus, in the perfect-fluid limit, where we assume that the fluid is always in local thermal equilibrium, no entropy production occurs and thus all **dissipative effects** are neglected.

So for perfect-fluid motion we necessarily have

$$\partial_\mu S^\mu = 0. \quad (1.7.19)$$

From (1.6.13) and (1.7.2) we find

$$S^\mu = (1 - \alpha)J^\mu + \beta T^{\mu\nu}u_\nu = \frac{\epsilon + nT - \mu n}{T}u^\mu \stackrel{(1.6.23)}{=} \frac{\epsilon + P - \mu n}{T}u^\mu \stackrel{(1.6.25)}{=} su^\mu. \quad (1.7.20)$$

Thus from (1.7.19) we find

$$Ds = -s\partial_\mu u^\mu = -s\nabla_\mu u^\mu. \quad (1.7.21)$$

From this we find for the **entropy per particle**

$$D\left(\frac{s}{n}\right) \stackrel{(1.7.10)}{=} \frac{Ds}{n} - \frac{s}{n^2}Dn = \frac{Ds}{n} + \frac{s}{n}\nabla_\mu u^\mu \stackrel{(1.7.21)}{=} 0. \quad (1.7.22)$$

Both (1.7.19) and (1.7.22) imply that the total entropy is conserved for an ideal fluid.

Finally we consider **sound-wave solutions**, i.e., we assume that the fluid is nearly at rest and in *global thermal equilibrium*, i.e., we assume

$$P(x) = P_0 + \tilde{P}(x), \quad s(x) = s_0 + \tilde{s}(x), \quad \vec{v}(x) = \tilde{\vec{v}}(x) \quad (1.7.23)$$

and that we can neglect all products of the small quantities with a tilde in the equations of motion.

In this limit the 1<sup>st</sup> Eq. (1.7.14) reads

$$n_0 b_0 \partial_t \tilde{\vec{v}} = -\vec{\nabla} \tilde{P} \quad (1.7.24)$$

and (1.7.10)

$$\partial_t \tilde{n} + n_0 \vec{\nabla} \cdot \tilde{\vec{v}} = 0. \quad (1.7.25)$$

In (1.7.24) and (1.7.25) we can interpret  $\vec{\nabla}$  as the usual gradient. To find a closed equation for the pressure we take the divergence of (1.7.24) and the time derivative of (1.7.25). This gives

$$\frac{1}{b_0} \partial_t^2 \tilde{n} = -\Delta \tilde{P}. \quad (1.7.26)$$

To close this equation we write

$$\tilde{n} = \left( \frac{\partial n_0}{\partial P_0} \right)_s \tilde{P}, \quad (1.7.27)$$

which holds in linear approximation. Since, as we have shown above, the total entropy is conserved we have to take the derivative of the equation of state at constant entropy. Then we can write (1.7.27) in the form

$$\frac{1}{v_s^2} \partial_t^2 \tilde{P} + \Delta \tilde{P} = 0 \quad (1.7.28)$$



with the **adiabatic speed of sound**

$$v_s = \sqrt{\frac{1}{h_0} \left( \frac{\partial P_0}{\partial n_0} \right)_s}. \quad (1.7.29)$$

In the non-relativistic limit we can set  $h_0 \simeq m$  and we find the well-known formula

$$v_s \simeq \sqrt{\left( \frac{\partial P_0}{\partial \varrho_0} \right)_s}, \quad (1.7.30)$$

where  $\varrho_0$  is the **mass density** of the fluid in global equilibrium.

For the application of **ultrarelativistic heavy-ion collisions** one makes ample use of this formalism of the perfect-fluid approximation. The idea is that after a surprising short time of  $t_{\text{fo}} \simeq 1 \text{ fm}/c$  after the collision the medium can be treated approximately as a (nearly) perfect fluid. As shown in the introduction, at RHIC and LHC (and to less accuracy also at the SPS), such a model gives a good description of a lot of hadronic transverse-momentum ( $p_T$ ) spectra and elliptic flow  $v_2$  for low  $p_T$ .

To switch back from the fluid dynamics with its macroscopic fields given by  $n$ ,  $P$ ,  $\epsilon$ ,  $s$ , and  $\vec{u}$ , one makes use of the fact that the phase-space distribution function is given by the local-equilibrium Maxwell-Jüttner distribution function (1.6.7) and switches back to a particle picture for the various hadrons measured in the detectors. This switch is a non-trivial issue. The basic idea is that we observe momentum spectra of particles which are determined by the moment, when the medium has become so dilute through the expansion and cooling of the fireball that all **elastic scatterings** have ceased. This is called the **thermal freeze-out**. Even before, i.e., at higher temperatures and densities already the **inelastic scatterings** have ceased. At this **chemical freeze-out** all the particle abundancies are fixed (except for decays of heavier resonance states to lighter stable particles). From **thermal models** one finds that the chemical freeze-out occurs pretty close to the phase transition at the pseudo-critical temperature  $T_c \simeq 160 \text{ MeV}$ . To determine temperature and chemical potentials at the thermal freeze-out one needs an **equation of state**, which is taken from the **thermal lattice-QCD calculations** and can be parametrized well by a hadron-resonance gas in the hadronic phase and a Quark-Gluon Plasma with effective in-medium (temperature dependent) masses above  $T_c$  with some interpolating smooth function in the region around  $T_c$ , describing the pretty rapid cross-over transition from the partonic to the hadronic phase.

As soon as one has determined the chemical potentials and temperature at an assumed thermal-freeze-out point (given by some energy density or temperature of the fluid cells), one can use the local-equilibrium Boltzmann-Jüttner distribution function to evaluate the momentum spectra of the particles. This is done in the **Cooper-Frye freeze-out description** [CF74]. To that end one uses the freeze-out criterion (e.g.,  $\epsilon(x) < \epsilon_{\text{fo}}$ ) to determine the **three-dimensional hypersurface** in Minkowski space, where the particles in the corresponding fluid elements decouple from the collective motion of the fluid and start freely streaming to the detectors. The volume element in the corresponding integral is characterized by a **covariant hypersurface “normal” vector**. If the hypersurface is parametrized by some generalized coordinates  $q^j$  ( $j \in \{1, 2, 3\}$ ) this hypersurface normal vector is given by

$$d^3 \sigma^\mu = d^3 q \epsilon^{\mu\nu\rho\sigma} \frac{\partial x_\nu}{\partial q^1} \frac{\partial x_\rho}{\partial q^2} \frac{\partial x_\sigma}{\partial q^3}, \quad (1.7.31)$$

where

$$\epsilon^{\mu\nu\rho\sigma} = \text{sign}[(\mu, \nu, \rho, \sigma)] \quad (1.7.32)$$

is the totally antisymmetric **Levi-Civita tensor**. Since  $\det \eta = -1$  the covariant components obey

$$\epsilon_{\mu\nu\rho\sigma} = -\epsilon^{\mu\nu\rho\sigma}, \quad (1.7.33)$$

and since for proper orthochronous Lorentz transformations  $\det \Lambda = 1$  these are tensor components of an invariant tensor of 4<sup>th</sup> rank under such transformations, which makes (1.7.31) a Lorentz vector.

Then the momentum distribution of the frozen-out hadrons is given by

$$\frac{dN}{d^3\vec{p}} = \frac{g}{(2\pi\hbar)^3} \int_{V_{\text{fo}}} d^3\sigma^\mu \frac{p_\mu}{E} f_{\text{eq}}(x, p) \quad (1.7.34)$$

with the Boltzmann-Jüttner distribution function

$$f_{\text{eq}}(x, p) = \exp\left[-\frac{p \cdot u(x) - \mu(x)}{T(x)}\right], \quad (1.7.35)$$

where  $p$  is the on-shell four-momentum of the hadron species under consideration and  $u(x)$  is the fluid four-flow velocity field as determined from the hydrodynamical simulation.

For further reading on the application of relativistic perfect-fluid dynamics in the description of heavy-ion collisions many review articles are available, e.g., [KH03, Hei04, Oll08].

## 1.8 The Boltzmann-Uehling-Uhlenbeck Equation

So far we have not taken into account the **quantum statistics** of **indistinguishable particles** in our transport equation. In this Section we shall give a semiclassical argument to take quantum effects due to **Bose-Einstein** and **Fermi-Dirac statistics** into account [UU33, Ueh34]. We shall give a more formal foundation from quantum field theory, as the most convenient description of many-body systems on a fundamental level in the next Chapter. Even before the discovery of the “modern quantum theory” the importance of the right statistical account of the occupation numbers of single-particle states, making a “macro state” by taking the average of these numbers, has been seen by the pioneers of “early quantum theory”. So, Einstein has shown that the **Planck distribution**, describing **black-body radiation** can be derived under the assumption of a kind of particle structure of electromagnetic waves, if one assumes that these “light quanta” or **photons** are emitted and absorbed in quantized energy portions  $\hbar\omega$ , where  $\omega$  is the frequency of the corresponding plane-wave field and that each photon carries a momentum  $\hbar\vec{k}$ . However, to obtain the correct Planck distribution, Einstein had to assume also the possibility of **spontaneous emission**, which takes into account that photons as particle-like entities we call **bosons** prefer to occupy states that are already occupied by other particles of the same kind.

Contrary to that other particles are **fermions** of which only one particle can occupy a single-particle state. In other words, if a single-particle state is already occupied by a fermionic particle, this state is blocked for all other fermions with the same quantum number. This is the **Pauli exclusion principle**, which is, e.g., crucial for an understanding of the shell structure of multi-electron atoms and the specific heat and electric conductivity of metals as described by the **Drude model** of the conduction electrons as a nearly ideal gas moving in the positively charged background of the crystal lattice making up the metal.

As we shall see in a moment, the so far considered **classical statistics** becomes valid as an approximation for both bosons and fermions, if the average occupation number of each single-particle states is small. The argument, how to modify the Boltzmann equation to take into account the “quantum indistinguishability” of particles is somewhat simpler for fermions. So we start with this case. Again, our aim is to derive an equation for the time evolution of the **single-particle phase-space distribution** due to the drift of the particles (also under the influence of an external force), represented by the left-hand side of the Boltzmann equation (1.3.41). The derivation of this part shows that this is unaffected by

the mentioned quantum effects, because it simply describes the time derivative of the particle numbers expressed in terms of the single-particle phase-space distribution.

On the other hand in the derivation of the **collision term** on the right-hand side we tacitly made the assumption that in a collision process it is not important, whether the phase-space cell of size  $d^6\xi = d^3\vec{x}d^3\vec{p}$  of the final state is already occupied by (one or more) particles or not. Now, contrary to this assumption, the Pauli exclusion principle tells us that a scattering process is not possible, if the final state is already occupied by other particles of the same kind. As we have seen in the derivation of (1.5.1), the number of single-particle states corresponding to the phase-space element  $d^6\xi$  is given by

$$G_j = \frac{g d^6\xi_j}{(2\pi\hbar)^3}, \quad (1.8.1)$$

where  $g$  is the degeneracy factor due to the spin-degrees of freedom<sup>12</sup>.

In the collision term for fermions we have to take the **Pauli exclusion principle** into account, according to which a scattering can only occur, if the single-particle final states are not occupied. This in the collision term we have to use the **available number of final states**, i.e.,

$$G_j^{(\text{avail})} = G_j - \frac{g}{(2\pi\hbar)^3} f d^6\xi_j = \frac{g d^6\xi_j}{(2\pi\hbar)^3} (1-f). \quad (1.8.2)$$

Thus we have to multiply the collision term by the corresponding **Pauli-blocking factors**,  $(1-f)$ , for the particles in the final state.

In this way we obtain the **Boltzmann-Uehling-Uhlenbeck (BUU) equations for fermions**

$$p^\mu \frac{\partial f}{\partial x^\mu} + m \frac{\partial (K^\mu f)}{\partial p^\mu} = \frac{1}{2} \frac{g}{(2\pi\hbar)^3} \int_{\mathbb{R}^3} \frac{d^3\vec{p}_2}{E_2} \int_{\mathbb{R}^3} \frac{d^3\vec{p}'_1}{E'_1} \int_{\mathbb{R}^3} \frac{d^3\vec{p}'_2}{E'_2} W(p'_1, p'_2 \leftarrow p_1, p_2) \times (f'_1 f'_2 \bar{f} \bar{f}_2 - f f_1 \bar{f}'_1 \bar{f}'_2). \quad (1.8.3)$$

Here we have introduced the abbreviation

$$\bar{f}(x, p) = 1 - f(x, p) \quad (\text{for fermions}). \quad (1.8.4)$$

The BUU equation (1.8.3) also holds for **bosons**. We only have to redefine (1.8.4) to

$$\bar{f}(x, p) = 1 + f(x, p) \quad (\text{for bosons}) \quad (1.8.5)$$

to take into account **Bose enhancement** as opposed to Pauli blocking in (1.8.4).

All the general steps in analyzing the physical meaning of the BUU equation are analogous to that given in the previous Sects. for the classical case. The main difference is the definition of **entropy**, where we have to take into account **Fermi-Dirac or Bose-Einstein statistics**.

We start with the case of **fermions** and follow the arguments in Sect. 1.5 taking the Pauli exclusion principle into account. So let  $G_j$  be the number of single-particle states in the phase-space volume element  $d^6\xi_j$  and  $N_j$  the number of particles occupying this state. Now due to the Pauli exclusion principle we can have at most one particle in a single-particle state, and the number of many-body

<sup>12</sup> $g = 2s + 1$  for massive particles of spin  $s \in \{0, 1/2, 1, \dots\}$ ,  $g = 1$  for massless scalar particles, and  $g = 2$  for massless particles with spin  $s \in \{1/2, 1, \dots\}$

states with  $N_j$  particles in the  $j$ th phase-space cell is given by the number of selecting  $N_j$  of the  $G_j$  states. Of course, for fermions we must have  $N_j \leq G_j$ . This number of combinations is given by

$$\Gamma_j = \binom{G_j}{N_j} = \frac{G_j!}{N_j!(G_j - N_j)!}. \quad (1.8.6)$$

Using the approximate Stirling formula for  $\ln N!$  for large  $N$  we find for the total entropy for a given distribution of particles  $N_j$

$$S = \sum_j [G_j \ln G_j - N_j \ln N_j - (G_j - N_j) \ln(G_j - N_j)]. \quad (1.8.7)$$

Introducing now the mean occupation number  $f_j = N_j/G_j$  of the  $j$ th phase-space cell, we find

$$S = - \sum_j G_j [f_j \ln f_j + (1 - f_j) \ln(1 - f_j)] = - \frac{g}{(2\pi\hbar)^3} \int_{\mathbb{R}^3} d^3\vec{x} \int_{\mathbb{R}^3} d^3\vec{p} [f \ln f + (1 - f) \ln(1 - f)]. \quad (1.8.8)$$

For bosons each quantum state can contain an arbitrary number of particles, but we cannot distinguish which individual particle occupies a given phase-space cell. This means that the number of many-body states given the distribution  $N_j$  over the phase-space cells is given by the number of ways to distribute  $N_j$  particles over the  $G_j$  states, which is given by<sup>13</sup>

$$\Gamma_j = \binom{G_j + N_j - 1}{N_j} = \frac{(G_j + N_j - 1)!}{(G_j - 1)!N_j!}. \quad (1.8.9)$$

The entropy then is given by

$$\begin{aligned} S &= \sum_j [(G_j + N_j) \ln(G_j + N_j) - N_j \ln N_j - G_j \ln G_j] \\ &= \frac{g}{(2\pi\hbar)^3} \int_{\mathbb{R}^3} d^3\vec{x} \int_{\mathbb{R}^3} d^3\vec{p} [(1 + f) \ln(1 + f) - f \ln f], \end{aligned} \quad (1.8.10)$$

where again we have used the Sterling approximation for factorials. We note that we can combine (1.8.8) and (1.8.10) into one equation for the **entropy four-current vector** by writing

$$S^\mu(x) = \frac{g}{(2\pi\hbar)^3} \int_{\mathbb{R}^3} d^3\vec{p} \frac{p^\mu}{E} [\xi(f + \xi) \ln(1 + \xi f) - f \ln f] \quad (1.8.11)$$

with

$$\xi = \begin{cases} 1 & \text{for bosons,} \\ -1 & \text{for fermions.} \end{cases} \quad (1.8.12)$$

We also see that we lead back to the Boltzmann equation and the expression for the entropy four-current if  $f \ll 1$ , i.e., if the average occupation number per phase-space volume is small compared to the number of single-particle states in this phase-space volume.

<sup>13</sup>This can be understood as follows: A distribution of  $N_j$  particles over  $G_j$  states can be symbolically noted by drawing  $(G_j - 1)$  vertical lines, giving  $G_j$  columns of a table, numbering the  $G_j$  individual single-particle states available in the  $j$ th phase-space cell. A possible distribution of  $N_j$  particles over these single-particle states is then given by making as many marks in each column as particles are contained in it. Thus the possibilities are given by the number of arrangements of  $N_j$  marks and  $G_j - 1$  vertical lines indicating the columns of our table, and this number is given by the binomial expression (1.8.9).

The derivation of the  $H$  theorem is now fully analogous to the one given in Sect. 1.5. The only difference is that, with the modified collision term in (1.8.3) and entropy four-current density (1.8.11) it turns out that for the equilibrium distribution, which leads to a stationary entropy, instead of (1.6.2) the expression

$$\phi = -\ln\left(\frac{f}{1 + \xi f}\right) \quad (1.8.13)$$

is a summational invariant. This immediately leads to (*Exercise!*)

$$\phi = \beta(u_\mu p^\mu - \mu) \Rightarrow f(x, p) = \frac{1}{\exp[\beta(x)(p \cdot u(x) - \mu(x))] - \xi}. \quad (1.8.14)$$

We note that for bosons, we are not free to choose the chemical potential,  $\mu$ , arbitrarily anymore. To understand this, we go to the local rest frame of a fluid cell, i.e.,  $u = (1, 0, 0, 0)$ . Then we see that  $p \cdot u - \mu = E^* - \mu = \sqrt{m^2 + \vec{p}^{*2}} - \mu \geq 0$  should hold in order to avoid a non-integrable singularity in  $F$  in (1.8.14). This implies that the chemical potential is constrained to

$$\mu \leq m \quad \text{for bosons.} \quad (1.8.15)$$

The physical reason for this is the following: Suppose we put a gas in a container and cool it down, keeping the total number of particles fixed. Below a certain temperature, we cannot keep the particle number fixed, because we cannot enhance  $\mu$  further above the mass of the particles due to the constraint (1.8.15). The resolution of this apparent paradox is that below that **critical temperature a phase transition** occurs, where the single-particle “ground state” ( $\vec{p} = 0$ ) becomes occupied by a macroscopic number of particles. These particles are collectively described by the corresponding single-particle ground-state wave function. This phenomenon is known as **Bose-Einstein condensation**.

In general for a relativistic gas, where particle creation and annihilation processes occur, we have to consider for each particle also its anti-particle, i.e., we obtain two coupled Boltzmann equations for the corresponding distribution functions. Then the particle number of particles or anti-particles is not conserved, but often there is a **conserved charge-like quantity** as electric charge or baryon number. Then the chemical potential refers to this conserved charge, and the anti-particles have the opposite charge of particles and thus in local or global equilibrium the chemical potential of the anti-particles are

$$\mu_{\bar{p}} = -\mu_p =: -\mu. \quad (1.8.16)$$

Thus the equilibrium distributions for particles and anti-particles in the local rest frame are given by

$$f_p^* = \frac{1}{\exp[\beta(E^* - \mu)] - \xi}, \quad f_{\bar{p}}^* = \frac{1}{\exp[\beta(E^* + \mu)] - \xi}. \quad (1.8.17)$$

Since the masses of the particles and antiparticles is identical,  $m_p = m_{\bar{p}} = m$ , the constraint for the chemical potential for bosons ( $\xi = +1$ ) thus is given by

$$-m \leq \mu \leq m \quad (\text{for bosons}). \quad (1.8.18)$$

In passing we note that Bose-Einstein condensation can be understood as an example of spontaneous symmetry breaking in many-body quantum-field theory. For more details, see, e.g., [Kap81, KG06].

## 1.9 The Modified Bessel Functions

We define the modified Bessel functions as the integrals

$$K_m(z) = \int_0^\infty dy \cosh(my) \exp(-z \cosh y). \quad (1.9.1)$$

First we derive a recursion relation:

$$K_{m+1}(z) - K_{m-1}(z) = \frac{2m}{z} K_m(z) \quad (1.9.2)$$

This is shown by integrating (1.9.1) by parts, which gives

$$\begin{aligned} K_m(z) &= \frac{z}{m} \int_0^\infty dy \sinh(my) \sinh y \exp(-z \cosh y) \\ &= \frac{z}{2m} \int_0^\infty dy \{ \cosh[(m+1)y] - \cosh[(m-1)y] \} \\ &= \frac{z}{2m} [K_{m+1}(z) - K_{m-1}(z)]. \end{aligned} \quad (1.9.3)$$

In a similar way we find for the derivative of the Bessel functions

$$\begin{aligned} \frac{d}{dz} K_m(z) &= - \int_0^\infty dy \cosh y \cosh(my) \exp(-z \cosh y) \\ &= - \frac{1}{2} \int_0^\infty dy \{ \cosh[(m+1)y] + \cosh[(m-1)y] \} \exp(-z \cosh y) \\ &= - \frac{1}{2} [K_{m+1}(z) + K_{m-1}(z)] \\ &\stackrel{(1.9.3)}{=} - \frac{mK_m(z) + zK_{m-1}}{z}. \end{aligned} \quad (1.9.4)$$

Further we need the behavior of the functions for  $z \gg 1$ . To find the asymptotic behavior for  $z \rightarrow \infty$  we can use the saddle-point approximation of the defining integral (1.9.1). To that end one writes the integrand in the form

$$\begin{aligned} \cosh(my) \exp(-z \cosh y) &= \exp \left[ -z \left( 1 + \frac{y^2}{2} \right) \right] \cosh(my) \exp \left[ -z \left( \cosh y - 1 - \frac{y^2}{2} \right) \right] \\ &= \exp \left[ -z \left( 1 + \frac{y^2}{2} \right) \right] \left[ 1 + \frac{m}{2} y^2 + \frac{m^4 - z}{24} y^4 + \mathcal{O}(y^6) \right] \end{aligned} \quad (1.9.5)$$

Plugging this into (1.9.1) we find the first two terms of the asymptotic expansion

$$K_m(z) \underset{z \rightarrow \infty}{\cong} \sqrt{\frac{\pi}{2z}} \exp(-z) \left[ 1 + \frac{4m^2 - 1}{8z} + \mathcal{O}\left(\frac{1}{z^2}\right) \right]. \quad (1.9.6)$$

# Bibliography

- [AS10] A. Altland and B. Simons, *Condensed Matter Field Theory*, Cambridge University Press, Cambridge, New York, Melbourne, Madrid, Cape Town, 2 edn. (2010).
- [Bay62] G. Baym, Self-Consistent Approximations in Many-Body Systems, *Phys. Rev.* **127**, 1391 (1962), <http://link.aps.org/abstract/PR/v127/i4/p1391>.
- [BGG<sup>+</sup>12] O. Buss, T. Gaitanos, K. Gallmeister et al., Transport-theoretical Description of Nuclear Reactions, *Phys. Rept.* **512**, 1 (2012), <http://dx.doi.org/10.1016/j.physrep.2011.12.001>.
- [BK61] G. Baym and L. Kadanoff, Conservation Laws and Correlation Functions, *Phys. Rev.* **124**, 287 (1961), <http://link.aps.org/abstract/PR/v124/i2/p287>.
- [BM90] W. Botermans and R. Malfliet, Quantum transport theory of nuclear matter, *Phys. Rept.* **198**, 115 (1990), [http://dx.doi.org/10.1016/0370-1573\(90\)90174-Z](http://dx.doi.org/10.1016/0370-1573(90)90174-Z).
- [Cas09] W. Cassing, From Kadanoff-Baym dynamics to off-shell parton transport, *Eur. Phys. J. ST* **168**, 3 (2009), 0808.0715, <http://dx.doi.org/10.1140/epjst>.
- [CF74] F. Cooper and G. Frye, Single-particle distribution in the hydrodynamic and statistical thermodynamics models of multiparticle production, *Phys. Rev. D* **10**, 186 (1974), <http://dx.doi.org/10.1103/PhysRevD.10.186>.
- [CK02] C. Cercignani and G. M. Kremer, *The relativistic Boltzmann Equation: Theory and Applications*, Springer, Basel (2002), <http://dx.doi.org/10.1007/978-3-0348-8165-4>.
- [Dan84a] P. Danielewicz, Quantum Theory of Nonequilibrium Processes I, *Ann. Phys.* **152**, 239 (1984), [http://dx.doi.org/10.1016/0003-4916\(84\)90092-7](http://dx.doi.org/10.1016/0003-4916(84)90092-7).
- [Dan84b] P. Danielewicz, Quantum Theory of Nonequilibrium Processes II. Application to Nuclear Collisions, *Ann. Phys.* **152**, 305 (1984), [http://dx.doi.org/10.1016/0003-4916\(84\)90093-9](http://dx.doi.org/10.1016/0003-4916(84)90093-9).
- [dvv80] S. R. de Groot, W. A. van Leeuwen and C. G. van Weert, *Relativistic kinetic theory: principles and applications*, North-Holland (1980).
- [FW71] A. L. Fetter and J. D. Walecka, *Quantum Theory of Many-Particle Systems*, McGraw-Hill Book Company, New York (1971).
- [Hee13] H. van Hees, Nonequilibrium relativistic many-body theory (2013), <http://fias.uni-frankfurt.de/~hees/publ/off-eq-qft.pdf>.

## BIBLIOGRAPHY

- [Hei04] U. W. Heinz, Concepts of heavy ion physics 165–238 (2004), hep-ph/0407360, <http://arxiv.org/abs/hep-ph/0407360>.
- [Kap81] J. I. Kapusta, Bose-Einstein Condensation, Spontaneous Symmetry Breaking, and Gauge Theories, Phys. Rev. D **24**, 426 (1981), <http://link.aps.org/doi/10.1103/PhysRevD.24.426>.
- [KB61] L. Kadanoff and G. Baym, *Quantum Statistical Mechanics*, A Lecture Note and Preprint Series, The Benjamin/Cummings Publishing Company, New York (1961).
- [Kel64] L. Keldysh, Diagram Technique for Nonequilibrium Processes, Zh. Eksp. Teor. Fiz. **47**, 1515 (1964), [Sov. Phys JETP **20** 1965 1018], <http://www.jetp.ac.ru/cgi-bin/e/index/e/20/4/p1018?a=list>.
- [KG06] J. I. Kapusta and C. Gale, *Finite-Temperature Field Theory; Principles and Applications*, Cambridge University Press, 2 edn. (2006).
- [KH03] P. F. Kolb and U. Heinz, Hydrodynamic description of ultrarelativistic heavy ion collisions (2003), published in R. C. Hwa, X.-N. Wang (Ed.), Quark Gluon Plasma 3, World Scientific.
- [KIV01] J. Knoll, Y. B. Ivanov and D. N. Voskresensky, Exact conservation laws of the gradient expanded Kadanoff-Baym equations, Ann. Phys. **293**, 126 (2001), <http://dx.doi.org/10.1006/aphy.2001.6185>.
- [LeB96] M. LeBellac, *Thermal Field Theory*, Cambridge University Press, Cambridge, New York, Melbourne (1996).
- [LP81] E. Lifshitz and L. Pitaevsii, *Physical Kinetics*, Pergamon Press, Oxford (1981).
- [O<sup>+</sup>14] K. Olive et al. (Particle Data Group), Review of Particle Physics, Chin. Phys. C **38**, 090001 (2014), <http://dx.doi.org/10.1088/1674-1137/38/9/090001>.
- [Oll08] J.-Y. Ollitrault, Relativistic hydrodynamics for heavy-ion collisions, Eur. J. Phys. **29**, 275 (2008), <http://dx.doi.org/10.1088/0143-0807/29/2/010>.
- [Ram07] J. Rammer, *Quantum Field Theory of Non-equilibrium States*, Cambridge University Press, Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, Sao Paulo (2007).
- [Sch61] J. Schwinger, Brownian Motion of a Quantum Oscillator, J. Math. Phys **2**, 407 (1961), <http://link.aip.org/link/doi/10.1063/1.1703727>.
- [Ueh34] E. A. Uehling, Transport Phenomena in Einstein-Bose and Fermi-Dirac Gases. II, Phys. Rev. **46**, 917 (1934), <http://dx.doi.org/10.1103/PhysRev.46.917>.
- [UU33] E. A. Uehling and G. E. Uhlenbeck, Transport Phenomena in Einstein-Bose and Fermi-Dirac Gases. I, Phys. Rev. **43**, 552 (1933), <http://dx.doi.org/10.1103/PhysRev.43.552>.