Quantum Mechanics, Local Realistic Theories, and Lorentz-Invariant Realistic Theories

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First, we demonstrate Bell's theorem, without using inequalities, for an experiment with two particles. Then we show that, if we assume realism and we assume that the "elements of reality" corresponding to Lorentz-invariant observables are themselves Lorentz invariant, we can derive a contradiction with quantum mechanics.

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In 1965 Bell [1] demonstrated that quantum mechanics is not a local realistic theory. He did this by deriving a set of inequalities which must be satisfied by any local realistic theory and then showing that these inequalities are violated by quantum mechanics. More recently Greenberger, Horne, and Zeilinger (GHZ) have demonstrated Bell's theorem (i.e., that quantum mechanics is not a local realistic theory) by means of a direct contradiction without using inequalities [2,3] (see also Mermin [4]). This demonstration applies to quantum states with three or more particles. However, except in the limiting case where the number of possible settings of each local variable is allowed to tend to infinity (Hardy [5]), it is not possible to use the method of GHZ to demonstrate Bell's theorem for two-particle states. In this Letter we take a different approach and show, by considering a new gedanken experiment, that it is possible to demonstrate Bell's theorem by means of a direct contradiction (i.e., without the need of inequalities) using a two-particle state.

It is possible for a theory to be nonlocal and Lorentz invariant at the same time. However, when the de Broglie–Bohm approach is applied to relativistic quantum theories [6] we find that these theories, in addition to being nonlocal, are also not Lorentz invariant at the level of the hidden variables. Consequently, having established that all realistic interpretations of quantum mechanics must be nonlocal (this is Bell's theorem) it is natural to ask whether there is an analogous theorem which proves that they must also be non-Lorentz invariant. We find that, if the "elements of reality" corresponding to Lorentz-invariant observables are themselves Lorentz invariant, then Lorentz-invariant realistic interpretations of quantum mechanics are not possible.

The gedanken experiment we are going to consider consists of two Mach-Zehnder-type interferometers $MZ^{\pm}$, one for positrons (MZ^+) and one for electrons (MZ^-), arranged so that two paths overlap as shown in Fig. 1. Each interferometer $MZ^{\pm}$ has an input mode, $s^{\pm}$, two paths inside the interferometer, $u^{\pm}$ and $v^{\pm}$, and two output modes, $c^{\pm}$ and $d^{\pm}$. Taken separately each interferometer is arranged so that, due to destructive interference, no positrons or electrons will be detected at detector $D^{\pm}$ in output $d^{\pm}$. The beam splitters $BS2^{\pm}$ are removable. Now, a positron and an electron are created simultaneously and fed into their respective interferometers. The apparatus is arranged such that, if the positron takes path u^+ inside MZ^+ and the electron takes path u^- inside MZ^-, then the two particles will meet at point P and annihilate one another with a probability equal to 1. Expressing this mathematically we have

$$|u^+\rangle|u^-\rangle \rightarrow |\gamma\rangle , \tag{1}$$

where $|u^{\pm}\rangle$ is the state of the positron or electron traveling along path $u^{\pm}$ and $|\gamma\rangle$ is the state of the radition produced on annihilation. We will find that, as a consequence of this possible interaction between the two particles, it becomes possible for positrons and electrons to arrive at detectors $D^{\pm}$. This gedanken experiment is a modification of a gedanken experiment proposed by the author [7] to investigate empty waves and the latter is an extension of a gedanken experiment proposed by Elitzur and Vaidman [8] to demonstrate the possibility of interaction-free measurement.

The operation of $BS1^{\pm}$ is given by

$$|s^{\pm}\rangle \rightarrow (1/\sqrt{2})(i|u^{\pm}\rangle + |v^{\pm}\rangle) . \tag{2}$$

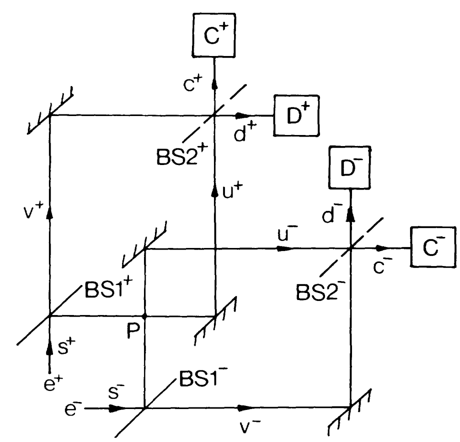

FIG. 1. Two Mach-Zehnder–type interferometers, one for positrons and one for electrons, arranged such that if a positron takes path u^+ and an electron takes path u^- then they will meet at point P and annihilate one another.

The operation of BS2$^{\pm}$ is given by

$$|u^{\pm}\rangle \rightarrow (1/\sqrt{2})(|c^{\pm}\rangle + i|d^{\pm}\rangle), \tag{3}$$

and

$$|v^{\pm}\rangle \rightarrow (1/\sqrt{2})(i|c^{\pm}\rangle + |d^{\pm}\rangle). \tag{4}$$

If BS2$^{\pm}$ is removed, then

$$|u^{\pm}\rangle \rightarrow |c^{\pm}\rangle, \tag{5}$$

and

$$|v^{\pm}\rangle \rightarrow |d^{\pm}\rangle \tag{6}$$

The initial state of the system is

$$|s^{+}\rangle|s^{-}\rangle. \tag{7}$$

After passing through the beam splitters BS1$^{\pm}$ this state evolves to [using (2)]

$$\tfrac{1}{2}(i|u^{+}\rangle + |v^{+}\rangle)(i|u^{-}\rangle + |v^{-}\rangle). \tag{8}$$

After passing point P the state becomes, using (1),

$$\tfrac{1}{2}(-|\gamma\rangle + i|u^{+}\rangle|v^{-}\rangle + i|v^{+}\rangle|u^{-}\rangle + |v^{+}\rangle|v^{-}\rangle). \tag{9}$$

If both BS2^{+} and BS2^{-} are removed then, using (5) and (6), we see that (9) evolves to the final state

$$\tfrac{1}{2}(-|\gamma\rangle + i|c^{+}\rangle|d^{-}\rangle + i|d^{+}\rangle|c^{-}\rangle + |d^{+}\rangle|d^{-}\rangle). \tag{10}$$

With BS2^{+} in place and BS2^{-} removed, using (3)-(6) we find that (9) evolves to the final state

$$\frac{1}{2\sqrt{2}}(-\sqrt{2}|\gamma\rangle - |c^{+}\rangle|c^{-}\rangle + 2i|c^{+}\rangle|d^{-}\rangle + i|d^{+}\rangle|c^{-}\rangle). \tag{11}$$

Similarly, with BS2^{+} removed and BS2^{-} in place we find that (9) evolves to the final state

$$\frac{1}{2\sqrt{2}}(-\sqrt{2}|\gamma\rangle - |c^{+}\rangle|c^{-}\rangle + i|c^{+}\rangle|d^{-}\rangle + 2i|d^{+}\rangle|c^{-}\rangle). \tag{12}$$

If both beam splitters BS2$^{\pm}$ are in place then using (3) and (4) we find that (9) evolves to the final state

$$\tfrac{1}{4}(-2|\gamma\rangle - 3|c^{+}\rangle|c^{-}\rangle + i|c^{+}\rangle|d^{-}\rangle + i|d^{+}\rangle|c^{-}\rangle - |d^{+}\rangle|d^{-}\rangle). \tag{13}$$

The notion of realism is introduced by letting the state of the positron-electron pair before measurements are made be described by hidden variables λ. These hidden variables can take different values each time the experiment is repeated. We can make two measurements on each particle—either with the beam splitter in place, denoted by 0, or with the beam splitter removed, denoted by ∞. The assumption of locality requires that the result of a measurement on one particle does not depend on the choice of measurement on the other particle. If the positron or electron is detected at $D^{\pm}$ with beam splitter BS2$^{\pm}$ in place then we will write $D^{\pm}(0,\lambda)=1$; if it is not detected then we will write $D^{\pm}(0,\lambda)=0$. If the positron or electron is detected at $C^{\pm}$ with the beam splitter BS2$^{\pm}$ removed then we will write $C^{\pm}(\infty,\lambda)=1$; if it is not detected then we will write $C^{\pm}(\infty,\lambda)=0$. In adopting this notation we have assumed locality because the result of a measurement on one particle does not depend on the choice of measurement made on the other particle. For example, $D^{+}(0,\lambda)$ does not depend on whether BS2^{-} is in place or not. We will now see that this leads to a contradiction with quantum mechanics. From (10) we see that

$$C^{+}(\infty,\lambda)C^{-}(\infty,\lambda)=0 \tag{14}$$

for every experiment because there is no $|c^{+}\rangle|c^{-}\rangle$ term. From (11) we see that

$$\text{if } D^{+}(0,\lambda)=1 \text{ then } C^{-}(\infty,\lambda)=1, \tag{15}$$

because if the positron is detected at D^{+} then the state is projected onto the last term in (11). Similarly, from (12) we see that

$$\text{if } D^{-}(0,\lambda)=1 \text{ then } C^{+}(\infty,\lambda)=1. \tag{16}$$

From (13) we see that

$$D^{+}(0,\lambda)D^{-}(0,\lambda)=1 \text{ for } \tfrac{1}{16}\text{th of experiments}. \tag{17}$$

Now consider an experiment for which $D^{+}(0,\lambda)D^{-}(0,\lambda)=1$. From (17) we see that this will happen in $\frac{1}{16}$th of the experiments. From (15) and (16) we see this implies that $C^{+}(\infty,\lambda)C^{-}(\infty,\lambda)=1$ for these experiments. However, (14) tells us that $C^{+}(\infty,\lambda)C^{-}(\infty,\lambda)=0$ for all experiments. Hence we have a contradiction between local realism and quantum mechanics. While this result can be compared to the GHZ result because no inequalities are used, it is dissimilar in that it only applies to $\frac{1}{16}$th of the experiments whereas the GHZ result applies to every experiment.

We now turn to the question of whether realistic theories can be Lorentz invariant. It is possible to have Lorentz-invariant theories which are nonlocal and therefore we will not assume locality in the following. If we are to speak about reality then we must say what we mean by reality. Instead of using the above approach of hidden variables we will adopt the following *sufficient condition for the element of physical reality* of Einstein, Podolsky, and Rosen [9] with some amendments due to Redhead (p. 72 of [10]): If we can predict with certainty (i.e., with probability equal to 1) the result of measuring a physical quantity, then there exists an element of reality corresponding to this physical quantity and having a value equal to the predicted measurement result. In the language of quantum mechanics this sufficient condition can be stated in the following way: If a system is in an eigenstate $|a\rangle$ of an operator $\hat{A}$, i.e., $\hat{A}|a\rangle = a|a\rangle$, then, even if we do not make a measurement, $[A]=a$, where $[A]$ is the value of the element of reality corresponding to the observable A. Some observables are Lorentz invariant. The value of such an observable *when it is measured* is frame independent both in quantum mechanics and in the classical analog. For example, the result of a *measurement* telling us whether or not a particle is inside a given box does not depend on the frame of reference of the observer. If the state of the particle is $|\text{in box}\rangle$ which corresponds to it being inside a box, then using the above reality condition we see that the statement "the particle is in the box" is an element of reality even if we do not make a measurement. We do not expect such elements of reality to depend on the frame of reference of the observer. Such statements are always Lorentz invariant in classical physics and we expect the same to be true in quantum mechanics. This motivates the following *necessary condition for Lorentz invariance of the elements of reality*: The value of an element of reality corresponding to a Lorentz-invariant observable is itself Lorentz invariant.

We will now see that these two conditions lead to a contradiction when applied to quantum mechanics. Consider the apparatus in Fig. 1 with the beam splitters $BS2^{\pm}$ in place. We will make use of the observables

$$\hat{U}^{\pm}=|u^{\pm}\rangle\langle u^{\pm}| . \tag{18}$$

Using the reality condition we obtain

$$\hat{U}^{\pm}|u^{\pm}\rangle=|u^{\pm}\rangle \Longrightarrow [U^{\pm}]=1 , \tag{19}$$

$$\hat{U}^{+}\hat{U}^{-}|u^{+}\rangle|u^{-}\rangle=|u^{+}\rangle|u^{-}\rangle \Longrightarrow [U^{+}U^{-}]=1 , \tag{20}$$

and

$$\hat{U}^{+}\hat{U}^{-}|u^{+},u^{-}\rangle_{\perp}=0 \Longrightarrow [U^{+}U^{-}]=0 , \tag{21}$$

where in each case the system is taken to be in the eigenstate shown in the equations on the left of the inferences (19) to (21) and where $|u^{+},u^{-}\rangle_{\perp}$ is any state vector orthogonal to $|u^{+}\rangle|u^{-}\rangle$. From the reality condition we have

$$\text{if } [U^{+}][U^{-}]=1 \text{ then } [U^{+}U^{-}]=1 . \tag{22}$$

We arrange the apparatus so that, in the laboratory frame of reference, the measurement on the positron and the measurement on the electron are simultaneous. However, we can consider a frame of reference F^{+} in which the measurement on the positron is made before the electron arrives at $BS2^{-}$. The state of the system just after the positron has passed through $BS2^{+}$ but before the electron has passed through $BS2^{-}$ can be obtained from (9) using (3) and (4):

$$\frac{1}{2\sqrt{2}}\left(-\sqrt{2}|\gamma\rangle-|c^{+}\rangle|u^{-}\rangle+2i|c^{+}\rangle|v^{-}\rangle+i|d^{+}\rangle|u^{-}\rangle\right) . \tag{23}$$

If the positron is detected at detector D^{+} still before the electron arrives at $BS2^{-}$, then the state is projected onto the last term in (23) and the state of the electron becomes $|u^{-}\rangle$. Therefore,

$$[U^{-}]=1 \text{ if detection at } D^{+} . \tag{24}$$

Now consider a frame of reference F^{-} in which the measurement on the electron occurs before the positron has reached $BS2^{+}$. By the symmetry of the apparatus we have, from (24),

$$[U^{+}]=1 \text{ if detection at } D^{-} . \tag{25}$$

Finally, consider the rest frame in which both measurements happen simultaneously. The state of the system before the particles go through $BS2^{\pm}$ is given by Eq. (9). However, this state is orthogonal to $|u^{+}\rangle|u^{-}\rangle$. Therefore,

$$[U^{+}U^{-}]=0 , \tag{26}$$

for all experiments.

The observables $U^{\pm}$ and $U^{+}U^{-}$ are Lorentz invariant and therefore if we adopt the condition for Lorentz invariance of elements of reality then the results (24)-(26) are truly independent of the frame of reference used in deriving them. Hence we can compare them. Consider an experiment in which there is a detection at D^{+} and D^{-}. From (13) we see that this will happen in $\frac{1}{16}$th of the experiments. For such experiments we obtain from (22), (24), and (25) the result

$$[U^{+}U^{-}]=1 , \tag{27}$$

but this contradicts (26), which is valid for all experiments. Therefore it is not possible to reproduce quantum mechanics with a realistic theory in which elements of reality corresponding to Lorentz-invariant observables are themselves Lorentz invariant. It is a simple matter to see that this result applies to any realistic interpretation which assumes that the particles have real trajectories —for example, the de Broglie-Bohm interpretation. Consider what happens if such an interpretation is used to calculate the trajectories in frame F^{+}. In this frame a detection at D^{+} requires that the electron has taken path u^{-} [from (24)]. If the electron takes path u^{-} then the positron must have taken path v^{+} otherwise it would have met the electron at point P and annihilated and would not then have been detected at D^{+}. Similar arguments apply in frame F^{-}. In this frame, if there is a detection at D^{-} then the positron must have taken path u^{+} [from (25)] and therefore, the electron must have taken path v^{-}. Consequently, if we consider a run of the experiment for which both $D^{+}=1$ and $D^{-}=1$, then the trajectories calculated in frame F^{+} contradict those calculated in frame F^{-}. The way out of this is to have a preferred frame of reference (thus violating Lorentz invariance). If the realistic interpretation is then applied in another frame then in some situations it will predict the "wrong" trajectories. However, it should be pointed out that, although the gedanken experiment we have discussed suggests a preferred frame of reference, it cannot be used to tell us which that preferred frame is.

One response to the above arguments might be to abandon realism. If we retain realism, however, then we are forced to accept that quantum mechanics implies both nonlocality and violation of Lorentz invariance. Nonlocality implies that, at the level of the hidden variables, there is faster than light transfer of information. This could lead to the possibility of sending information backward in time giving rise to well-known causal paradoxes. However, if there is a special frame of reference as is implied by the violation of Lorentz invariance then such causal paradoxes are blocked. Therefore, although nonlocality does not require a special frame of reference, it is most naturally incorporated into a theory in which there is a special frame of reference. One possible candidate for this special frame of reference is the one in which the cosmic background radiation is isotropic. However, other than the fact that a realistic interpretation of quantum mechanics requires a preferred frame and the cosmic background radiation provides us with one, there is no readily apparent reason why the two should be linked.

Since writing the first version of this article, I have become aware of an article by Clifton, Pagonis, and Pitowsky [11] which contains a similar discussion on Lorentz invariance to that above but in the context of the GHZ gedanken experiment. I would like to thank Partha Ghose, David Hind, Pan Kaloyerou, Gino Lepore, Michael Lochwood, Euan Squires, Anton Zeilinger, and Marek Żukowski for many hours of stimulating conversations on this and closely related topics. This work is funded by the Science and Engineering Research Council.

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