# SUMMING THE CURIOUS SERIES OF KEMPNER AND IRWIN 

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Abstract. In 1914, Kempner proved that the series

$$
1 / 1+1 / 2+\ldots+1 / 8+1 / 10+1 / 11+\ldots+1 / 18+1 / 20+1 / 21+\ldots
$$

where the denominators are the positive integers that do not contain the digit 9 , converges to a sum less than 90 . The actual sum is about 22.92068. In 1916, Irwin proved that the sum of $1 / n$ where n has at most a finite number of 9 's is also a convergent series. We show how to compute sums of Irwins' series to high precision. For example, the sum of the series

$$
1 / 9+1 / 19+1 / 29+1 / 39+1 / 49+\ldots
$$

where the denominators have exactly one 9 , is about 23.044287080747848 31968. Another example: the sum of $1 / \mathrm{n}$ where n has exactly 100 zeros is about $10 \ln 10+1.007 \times 10^{-197} \approx$ 23.02585; note that the first, and largest, term in this series is the tiny $1 /$ googol.

## 1. Introduction

The harmonic series

$$
\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\ldots
$$

diverges. The sum can be made as large as we wish just by adding enough terms. For example, we can make the sum exceed 10 by adding the first 12367 terms; we can make the sum exceed 100 by adding 15092688622113788323693563264538101449859497 terms [2], [12], [13].

But in a 1914 paper titled "A Curious Convergent Series", A. J. Kempner [9] proved that the series

$$
\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{8}+\frac{1}{10}+\frac{1}{11}+\cdots+\frac{1}{18}+\frac{1}{20}+\frac{1}{21}+\cdots
$$

where the denominators are the positive integers that do not contain the digit 9 , converges to a sum less than 90. At first glance, this is surprising because Kempner's series looks like the harmonic series with every $10^{\text {th }}$ term deleted. If that were truly the case, then Kempner's series would diverge. Kempner's result was indeed curious!

However, in the long run, Kempner's series thins out enough to allow it to converge. For example, the vast majority of integers having 100 digits will contain a 9 somewhere within them, so most 100-digit numbers do not occur as denominators in Kempner's "no 9" series.

The author first learned of this result from [6, pp. 120-121]. This topic has even made its way into popular books such as [7, pp. 31-34] and [17, p. 81].

The sum of Kempner's "no 9" series is about 22.92067661926415034816 [1]. However, the convergence is so slow that the sum of all $\approx 10^{28}$ terms with denominators less than $10^{30}$ is still less than 22 , so this series cannot be accurately summed by brute force addition of terms. The following generalization also holds: if we delete from the harmonic series all terms whose denominators have any set of numbers with one or more digits, the resulting series also converge. Schmelzer and Baillie [11] showed how to compute sums of these slowly-converging series. For example, they calculate that the sum of $1 / n$ where $n$ has no even digits, is about 3.1717654734 15905.

In 1916, Frank Irwin [8] generalized Kempner's result in a different way: he showed that the sum of $1 / n$ where $n$ has at most any fixed number of occurrences of one or more digits, also converges. It follows that the sum of $1 / n$ where $n$ has exactly $n_{1}$ occurrences of digit $d_{1}, n_{2}$ occurrences of $d_{2}$, etc., is a convergent series. Kempner's "no 9 " series is simply an Irwin series with one condition placed on the digits, namely: $n_{1}=0, d_{1}=9$.

Irwin's result was also curious. In fact, Irwin used the same title for his article that Kempner had used two years earlier.

At first glance, it might appear that Irwin's "one 9" series, which begins

$$
\frac{1}{9}+\frac{1}{19}+\frac{1}{29}+\ldots
$$

is just the harmonic series with $90 \%$ of its terms deleted. If that were the case, then this series would diverge. However, if you think of a random 100-digit integer, it almost certainly will not have exactly one 9 . Therefore, far more than $90 \%$ of $n$ 's with 100 -digit denominators are excluded from the "one 9" series. In fact, with such a restriction on a single digit, in the long run, the denominators thin out enough to make the series converge.

These series studied by Irwin also converge very slowly. So, a natural question is, "What are their sums?" It seems that, until now, no one has accurately computed sums of Irwins' series. This article shows how to do so.

For example, we will calculate that the sum of $1 / n$ where $n$ has exactly one 9 is about 23.04428 708074784831968.

So, we have these amusing results:

$$
\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\ldots
$$

diverges, but

$$
\frac{1}{1}+\cdots+\frac{1}{8}+\frac{1}{10}+\cdots+\frac{1}{18}+\frac{1}{20}+\cdots+\frac{1}{28}+\frac{1}{30}+\cdots \approx 22.92068
$$

and

$$
\frac{1}{9}+\frac{1}{19}+\frac{1}{29}+\cdots \approx 23.04429
$$

The total of these last two series is not the sum of the harmonic series! Why? Observe that $1 / 99$ is missing from both series. These last two series, combined, have denominators with at most one 9 , while the harmonic series has denominators with arbitrarily many 9 's.
Notice that the sum of Irwin's "one 9" series is larger than the sum of Kempner's "no 9" series, given above. This is somewhat surprising because the "one 9 " series starts off with smaller terms: it begins $1 / 9+1 / 19+1 / 29+1 / 39+\ldots$, while the "no 9 " series begins $1 / 1+1 / 2+\ldots+$ $1 / 8+1 / 10+\ldots$. As we will see later, the partial sums of the "one 9 " series remain less than the final sum of the "no 9 " series until we include some terms whose denominators have 70 digits. There are $N_{i}=9^{i-1}+8(i-1) 9^{i-2}$ integers with $i$ digits that have exactly one 9 , so the "one $9 "$ series needs more than $N_{1}+N_{2}+\cdots+N_{69} \approx 5 \times 10^{66}$ terms to exceed the sum of the "no 9 " series.

Here's a more elaborate example: the sum of $1 / n$ where $n$ has at most one 1 , two 2's, three 3's, four 4's and five 5's, with no conditions placed on the other digits, is about 27.5600829488 9636705754 . The sum where $n$ has exactly one 1, two 2's, three 3's, four 4's, and five 5's is about 0.0046539022540563815564.

The sum of $1 / n$ where $n$ has 100 zeros, begins with the miniscule term $1 /$ googol. However, we will see in Section 6 that the sum of this series is also larger than the sum of the "no 9 " series!

The trick to summing the series of both Kempner and Irwin is to recognize that the denominators in the series form patterns. Consider Kempner's sum of $1 / n$ where the denominators $n$ have no 5 . Let $S_{i}$ be the set of all such $n$ that have $i$ digits. Then we can form $S_{i+1}$ as follows:

$$
S_{i+1}=\bigcup_{x \in S_{i}}\{10 x, 10 x+1,10 x+2,10 x+3,10 x+4,10 x+6,10 x+7,10 x+8,10 x+9\}
$$

Each $x$ in $S_{i}$ gives rise to nine numbers in $S_{i+1}$. Furthermore, the reciprocals of the numbers in $S_{i+1}$ are about $1 / 10$ as large as those of the corresponding numbers in $S_{i}$. Therefore, the sum of reciprocals of elements of $S_{i+1}$ is about $9 / 10$ as large as the sum of reciprocals of elements of $S_{i}$. This argument, made a little more rigorous, explains why Kempner's "no 9" series converges. It also provides a rough bound on the sum of the series.

The next section shows how to perform the same trick with Irwins' series.
All numbers in this article are rounded in the last decimal place displayed unless they are followed by trailing dots. A statement like " $n$ has $k$ zeros", means " $n$ has exactly $k$ zeros" unless stated otherwise. Finally, we assume the base is 10 unless stated otherwise.

An earlier version of this article was posted to Math ArXiv in 2013. Links that no longer work have been removed from the bibliography.

For the reader's convenience, this new version (2015) includes the kempnerSums.m package to compute sums of Kempner's series. This code is located in Section 8, below. An earlier version
of this code was used to compute the results in [11]. See Section 3 for a brief description of this code.

## 2. The Algorithm

Irwins' series converge much too slowly to compute their sums by simply adding terms. It's easy to add terms having denominators up to, say, 7 digits, but we must use some type of extrapolation procedure to get much beyond that point. It turns out that we can use sums over denominators with $i$ digits to compute the needed sums over denominators with $i+1$ digits. Then we repeat the process. As $i$ increases, the sums over denominators with $i$ digits get smaller and smaller. We will explain the algorithm for when there are two conditions for $1 / n$ to be included in the series: that is, where $n$ has (exactly) $n_{1}$ occurrences of the digit $d_{1}$ and $n_{2}$ occurrences of the digit $d_{2}$. The idea extends to conditions on any number of digits, and easily generalizes to bases other than 10 .

Define $S\left(i, k_{1}, k_{2}\right)$ to be the set of positive integers with $i$ digits that have $k_{1}$ occurrences of digit $d_{1}$ and $k_{2}$ occurrences of $d_{2}$. We can generate the set $S\left(i+1, k_{1}, k_{2}\right)$ from $S\left(i, k_{1}-1, k_{2}\right)$, $S\left(i, k_{1}, k_{2}-1\right)$, and $S\left(i, k_{1}, k_{2}\right)$, in the following way:
(a) For each $x$ in $S\left(i, k_{1}-1, k_{2}\right)$ : multiply by 10 , then add $d_{1}$.
(b) For each $x$ in $S\left(i, k_{1}, k_{2}-1\right)$ : multiply by 10 , then add $d_{2}$.
(c) For each $x$ in $S\left(i, k_{1}, k_{2}\right)$ : multiply by 10 , then add $d=0,1,2, \ldots, 9$, except for $d_{1}$ and $d_{2}$.

Step (a) starts with an $i$-digit number having $k_{1}-1$ occurrences of $d_{1}$ and appends $d_{1}$ as the final digit, forming an $(i+1)$-digit number having $k_{1}$ occurrences of $d_{1}$. Step (b) does the same for $k_{2}$ and $d_{2}$. In step (c), the $i$-digit numbers already have the desired number of $d_{1}$ and $d_{2}$, so in this step, we create $(i+1)$-digit numbers by appending all digits except $d_{1}$ and $d_{2}$. Together, steps (a), (b), and (c) generate $S\left(i+1, k_{1}, k_{2}\right.$ ). If $k_{1}$ is 0 , we omit step (a). If $k_{2}$ is 0 , we omit step (b).

Next, define

$$
t\left(i, j, k_{1}, k_{2}\right)=\sum_{x \in S\left(i, k_{1}, k_{2}\right)} \frac{1}{x^{j}} .
$$

For $j=1,2, \ldots$, we will show how to compute $t\left(i+1, j, k_{1}, k_{2}\right)$ by using the values of $t\left(i, j, k_{1}-\right.$ $\left.1, k_{2}\right), t\left(i, j, k_{1}, k_{2}-1\right)$, and $t\left(i, j, k_{1}, k_{2}\right)$.

If $x$ is an $i$-digit number and $d$ is any digit, then the reciprocal of the $(i+1)$-digit number $10 x+d$ can be expanded in powers of $1 / x$ :

$$
\frac{1}{10 x+d}=\frac{1}{10 x\left(1+\frac{d}{10 x}\right)}=\frac{1}{10 x} \sum_{n=0}^{\infty}(-1)^{n}\left(\frac{d}{10 x}\right)^{n}=\sum_{n=0}^{\infty}(-1)^{n} \frac{d^{n}}{(10 x)^{n+1}} .
$$

A similar expansion holds for higher powers:

$$
\begin{aligned}
\frac{1}{(10 x+d)^{j}} & =\frac{1}{(10 x)^{j}\left(1+\frac{d}{10 x}\right)^{j}}=\frac{1}{(10 x)^{j}} \sum_{n=0}^{\infty}(-1)^{n} \frac{(j+n-1)!}{n!(j-1)!}\left(\frac{d}{10 x}\right)^{n} \\
& =\sum_{n=0}^{\infty}(-1)^{n} \frac{(j+n-1)!}{n!(j-1)!} \frac{d^{n}}{(10 x)^{j+n}}
\end{aligned}
$$

This expansion is the negative binomial series [16]. This series converges if $\left|\frac{d}{10 x}\right|<1$.
Now, recalling step (a), we sum these expansions for all $x$ in $S\left(i, k_{1}-1, k_{2}\right)$. Call this sum $A$.

$$
\begin{align*}
A= & \sum_{x \in S\left(i, k_{1}-1, k_{2}\right)} \frac{1}{\left(10 x+d_{1}\right)^{j}}=\sum_{x \in S\left(i, k_{1}-1, k_{2}\right)} \sum_{n=0}^{\infty}(-1)^{n} \frac{(j+n-1)!}{n!(j-1)!} \frac{d_{1}^{n}}{(10 x)^{j+n}}  \tag{2.1}\\
& =\sum_{n=0}^{\infty}(-1)^{n} \frac{(j+n-1)!}{n!(j-1)!} \frac{d_{1}^{n}}{10^{j+n}} \sum_{x \in S\left(i, k_{1}-1, k_{2}\right)} \frac{1}{x^{j+n}} .
\end{align*}
$$

We can rearrange the order of this series because it converges absolutely. To see this, note that the expansion of $1 /(10 x+d)^{j}$ but without the $(-1)^{n}$, is just the expansion of $1 /(10 x-d)^{j}$, which also converges.

Moreover, for a given $i$ and $j$, the sum

$$
\sum_{x \in S\left(i, k_{1}-1, k_{2}\right)} \frac{1}{x^{j+n}}
$$

approaches 0 rapidly as $n$ approaches infinity. In this sum, $x$ ranges over a subset of $i$-digit numbers. Each value of $x$ is at least $10^{i-1}$, so $1 / x \leq 1 / 10^{i-1}$. So,

$$
\frac{1}{x^{j+n}} \leq \frac{1}{10^{(i-1)(j+n)}}
$$

There are fewer than $10^{i}$ such numbers $x$, so the sum is less than

$$
\frac{10^{i}}{10^{(i-1)(j+n)}}=\frac{1}{\left(10^{i-1}\right)^{n}} \cdot \frac{10^{i+j}}{10^{i j}}
$$

Likewise, summing over the sets described in steps (b) and (c), we get

$$
\begin{equation*}
B=\sum_{x \in S\left(i, k_{1}, k_{2}-1\right)} \frac{1}{\left(10 x+d_{2}\right)^{j}}=\sum_{n=0}^{\infty}(-1)^{n} \frac{(j+n-1)!}{n!(j-1)!} \frac{d_{2}^{n}}{10^{j+n}} \sum_{x \in S\left(i, k_{1}, k_{2}-1\right)} \frac{1}{x^{j+n}} \tag{2.2}
\end{equation*}
$$

and

$$
\begin{align*}
C= & \sum_{d} \sum_{x \in S\left(i, k_{1}, k_{2}\right)} \frac{1}{(10 x+d)^{j}}  \tag{2.3}\\
& =\sum_{n=0}^{\infty}(-1)^{n} \frac{(j+n-1)!}{n!(j-1)!} \frac{0^{n}+1^{n}+\cdots+9^{n}-d_{1}^{n}-d_{2}^{n}}{10^{j+n}} \sum_{x \in S\left(i, k_{1}, k_{2}\right)} \frac{1}{x^{j+n}} .
\end{align*}
$$

The sum in (2.3) is over digits $d$ other than $d_{1}$ or $d_{2}$. The factor

$$
\frac{0^{n}+1^{n}+\cdots+9^{n}-d_{1}^{n}-d_{2}^{n}}{10^{j+n}}
$$

is less than

$$
\frac{0^{n}+1^{n}+\cdots+9^{n}}{10^{j+n}}<\frac{9^{n}+9^{n}+\cdots+9^{n}}{10^{j+n}}=\frac{10 \cdot 9^{n}}{10^{j+n}}=\left(\frac{9}{10}\right)^{n} \cdot \frac{1}{10^{j-1}}
$$

so for a fixed $j$, this, too, rapidly approaches 0 as $n$ approaches infinity.
Together, the sets generated by steps (a) - (c) form the set $S\left(i+1, k_{1}, k_{2}\right)$. Also, $A+B+C=$ $t\left(i+1, j, k_{1}, k_{2}\right)$. So, we have just computed a needed sum over $(i+1)$-digit numbers by using sums over $i$-digit numbers. Programming detail: in (2.3), we take $0^{0}=1$.

In order to compute $t\left(i+1, j, k_{1}, k_{2}\right)$, we used the values of $t\left(i, j, k_{1}-1, k_{2}\right)$ and $t\left(i, j, k_{1}, k_{2}-1\right)$. But $t\left(i, j, k_{1}-1, k_{2}\right)$, in turn, was computed using $t\left(i-1, j, k_{1}-2, k_{2}-1\right)$ and $t\left(i-1, j, k_{1}-\right.$ $\left.1, k_{2}-2\right)$. This means that, for each $i$ and $j$, in order to compute $t\left(i+1, j, n_{1}, n_{2}\right)$, we must compute all $\left(n_{1}+1\right)\left(n_{2}+1\right)$ values of $t\left(i, j, k_{1}, k_{2}\right)$ for $0 \leq k_{1} \leq n_{1}$ and $0 \leq k_{2} \leq n_{2}$.

If we set $j=1$ and add the $t\left(i, 1, n_{1}, n_{2}\right)$ values over all $i$, then we get the sum of $1 / n$ where $n$ has exactly $n_{1}$ occurrences of $d_{1}$ and exactly $n_{2}$ occurrences of $d_{2}$. If we add the $t\left(i, 1, k_{1}, k_{2}\right)$ values over all $i$, and over all $k_{1}$ and $k_{2}$ with $0 \leq k_{1} \leq n_{1}$ and $0 \leq k_{2} \leq n_{2}$, then we get the sum of $1 / n$ where $n$ has at most $n_{1}$ occurrences of $d_{1}$ and at most $n_{2}$ occurrences of $d_{2}$. This sum may also be of interest.

If there is one condition on the digits, say, $n_{1}$ occurrences of digit $d_{1}$, then the procedure is similar, except that we omit step (b) above, and ignore all terms involving $d_{2}$ in equation (2.3).

If there are three conditions on the digits $\left(n_{1}, d_{1}\right),\left(n_{2}, d_{2}\right),\left(n_{3}, d_{3}\right)$, the procedure is similar, except that we have three steps in place of (a) and (b) above, and that for each $i$ and $j$, we must compute $\left(n_{1}+1\right)\left(n_{2}+1\right)\left(n_{3}+1\right)$ values for the $t$ array. The time and memory requirements increase accordingly. Still, even on a personal computer, we can specify a condition for each of the ten digits, provided the product $\left(n_{1}+1\right)\left(n_{2}+1\right) \ldots\left(n_{10}+1\right)$ is not too large. See examples 1,2 , and 3 in Section 5.

To summarize, our calculation goes as follows. We start by calculating $t\left(i, j, k_{1}, k_{2}\right)$ for $i=1$, 2 , and 3 , by explicitly adding the terms whose denominators are in the sets $S\left(i, k_{1}, k_{2}\right)$, for all $0 \leq k_{1} \leq n_{1}$ and $0 \leq k_{2} \leq n_{2}$. For $i=1$ and 2 , we need only $j=1$. For $i=3$, we explicitly add terms to compute all sums $t\left(3, j, k_{1}, k_{2}\right)$ for $j \leq J$, where $J$ will depend on the number of
decimal places desired in the final answer. Then we begin to extrapolate using equations (2.1) - (2.3), along with the $t\left(3, j, k_{1}, k_{2}\right)$ values. We compute the needed $t\left(i, j, k_{1}, k_{2}\right)$ for $i=4$ and $j=1,2, \ldots, J$. We then use those values to compute $t\left(i, j, k_{1}, k_{2}\right)$ for $i=5$, etc. We continue until the $t\left(i, j, k_{1}, k_{2}\right)$ values become small enough to be neglected. Of course, the more decimal places we want, the larger the range of $i$ and $j$ values we will need to compute.

As we proceed through a calculation, for each $i$ we can print the partial sums over denominators having $i$ digits. This allows us to estimate how far we need to go in the series in order to make the partial sums exceed some desired threshold. For example, with the "one 9 " series mentioned in the Introduction, the partial sum through all denominators having $\leq 69$ digits is about 22.90872 . The partial sum over denominators having $\leq 70$ digits is about 22.92072, which (finally) exceeds $22.92067 \ldots$, the sum of Kempner's "no 9 " series.

Also, for all $i \leq 68$ (but not for $i=69$ ), the partial sum of the "one 9 " series over denominators having $\leq i$ digits is less than the corresponding partial sum of the "no 9 " series over denominators having $\leq i$ digits. However, note that this does not prove that every partial sum of the "one 9 " series up to that point is smaller, because we have calculated the partial sums only at powers of 10 .

## 3. Mathematica Implementations

A Mathematica package that implements the algorithm in Section 2 is given after the references (see Section 7). To use this code, you can copy and paste it directly into Mathematica. You can also save it in a package (.m) file (for example, irwinSums.m), and read that file into Mathematica with

```
<< irwinSums.m
```

iSum is the function that produced most of the results in this paper. The general input format for iSum is

```
iSum[ digits, counts for those digits, number of decimal places, base ],
```

where the last two parameters are optional. If they are omitted, they default to 15 and 10 , respectively.

For example, iSum [9, 1] computes the sum of $1 / n$ where $n$ has exactly one 9 . The output is

```
sum = 23.044287080747848
    sum with exactly 0 occurrences of 9 = 22.920676619264150
    sum with exactly 1 occurrence of 9 = 23.044287080747848
    sum with at most 1 occurrence of 9 = 45.964963700011999
```

The requested sum is 23.044287080747848 . As noted in Section 2, in order to compute the sum of $1 / n$ where $n$ has one 9 , we also required the sum where $n$ has no occurrence of 9 ; this result is also displayed. The last sum, 45.96..., is the sum of $1 / n$ where n has either zero or one 9.

The sum of $1 / n$ where $n$ has two 0 's and three 9 's is computed with iSum [\{0, 9$\},\{2,3\}]$. The output is

```
sum = 3.603022188537689
    sum with at most 2 0's and at most 3 9's = 55.428091573952601
```

The requested sum is $3.60302 \ldots$. Because conditions are placed on more than one digit, iSum does not print the sums of all $(2+1)(3+1)=12$ combinations where $n$ has
exactly 00 's and exactly 09 's
exactly 00 's and exactly 19
exactly 00 's and exactly 29 's
exactly 00 's and exactly 39 's
exactly 10 and exactly 09 's
exactly 20 's and exactly 3 9's.
Only the total of these intermediate results, $55.42809 \ldots$, is displayed.
Two other functions in the Mathematica package will be discussed later, in Section 6:
iPartialSum [k, d, p]
computes the partial sum of $1 / n$ where $n$ has $k$ occurrences of $d$, through denominators $n<10^{p}$.
iPartialSumThreshold[k, d, t]
tells how many digits are required in the denominators to make the partial sum exceed the threshold, t .

There is also a version of iSum which formats the output, breaking digits after the decimal point into groups of 5, for easy reading. This function is iSumFormatted, and takes the same parameters as iSum.

As mentioned in the Introduction, this paper now includes the kempnerSums.m package.
To use this code, you can copy and paste it directly into Mathematica. You can also save it in a package (.m) file (for example, kempnerSums.m) in an appropriate directory, then read that file into Mathematica with
<< kempnerSums.m
The primary function in this package is kSum [ ].
Certain sums can be computed with either package. For example, after reading in the appropriate package, the sum of $1 / n$ where $n$ has no 9 can be computed with either iSum [9, 0] or kSum [9]. Both calculations give 22.920676619264150 .

However, each package can compute things that the other can't. For example, with kSum [ ], you can compute the sum of $1 / n$ where $n$ has no occurrence of a string of digits. So, the sum
of $1 / n$ where $n$ has no ' 314159 ' in base 10 can be computed with $k S u m$ [314159]. The result is $2.302582333863782607892 \cdot 10^{6}=2302582.333863782607892$.

## 4. Confirming the Calculations

The algorithm in the Section 2 can be checked as follows. For a given set of conditions on the digits $\left(n_{1}, d_{1}\right),\left(n_{2}, d_{2}\right), \ldots$, we explicitly compute the sums through denominators of, say, 7 digits. Then, we explicitly compute the sums through, say, just 4 digits, and use equations (2.1) - (2.3) to estimate the sums for 5,6 , and 7 digits. In all cases tested, the estimated sums agree with the explicitly-computed sums.
Generally, the series considered in this paper converge very slowly. However, in a few special cases, they converge rapidly enough that we can compute their sums directly. These cases can serve as a check on the algorithm. For example, the sum of $1 / n$ where $n$ has no 0 in base 2 is

$$
\frac{1}{1}+\frac{1}{3}+\frac{1}{7}+\frac{1}{15}+\cdots=\sum_{n=1}^{\infty} \frac{1}{2^{n}-1}
$$

which converges rapidly to about 1.60669515241529176378 . An equivalent result is produced by

$$
\text { iSum }[0,0,15,2]
$$

which gives 1.6066951524 15292. Because we are specifying a non-default base of 2 in the fourth parameter, we must also specify the third parameter (15), the number of decimal places.
The algorithm presented above tells us that the sum of $1 / n$ where $n$ has a single 0 in base 2 is (to 25 decimals) iSum [0, 1, 25, 2] $=1.46259073504436469954$ 61454. Writing out the denominators in binary, one can easily compute that the (rapidly convergent) sum is

$$
\sum_{n=2}^{\infty} \sum_{k=0}^{n-2} \frac{1}{2^{n}-1-2^{k}} \approx 1.4625907350443646995461454
$$

Finally, when we calculate the sum of $1 / n$ where $n$ has one occurrence of a digit $d$, the algorithm necessarily computes the sum of $1 / n$ where $n$ has zero occurrences of $d$. When we do these calculations to 5000 decimals for each of the ten digits, the sums for zero occurrences match the corresponding 5000 decimal place sums obtained by the algorithm in [11].

## 5. Examples

Example 1 (a). Consider the sum of $1 / n$ where $n$ has exactly one of each decimal digit. Because we are limiting the number of occurrences of every digit, this series will have only a finite number of terms. In every denominator of this series, each digit occurs exactly once, so every denominator has exactly 10 digits, all distinct. There are $10!\times 9 / 10=3265920$ such numbers. The approximate sum of this finite series can be calculated to 20 decimals with
iSum $[\{0,1,2,3,4,5,6,7,8,9\},\{1,1,1,1,1,1,1,1,1,1\}, 20]$.
The output is

```
sum = .00082589034791925293861
    sum with at most 1 0, ..., and at most 1 9 = 8.92994817475544342417
```

Example 1 (b). In order obtain the sum in Example 1 (a), our algorithm first computes the $2^{10}$ sums over those $n$ which have all combinations of either zero or one occurrence of each of the ten digits. Together, these $n$ comprise the positive integers that have distinct digits. There are 8877690 of these integers from 1 through 9876543210 . Elementary problem E2533 in the American Mathematical Monthly [10] asks for the sum of their reciprocals. Example 1 (a) shows that this sum is about 8.92995, a more precise answer than the one given in [5]. Interestingly, Eric Weisstein [15] used Mathematica to compute the exact value of this finite sum, a fraction whose numerator and denominator have 14816583 and 14816582 digits, respectively.

Table 1 shows a similar calculation for bases 2 through 10 . For example, in base 3, there are four $n$ that have exactly one 0 , one 1 , and one 2 . These are $102_{3}=11,120_{3}=15,201_{3}=19$, and $210_{3}=21$. The sum of their reciprocals is $1886 / 7315 \approx 0.2578263841$. Our algorithm gives iSum $[\{0,1,2\},\{1,1,1\}, 20,3]=0.25782638414217361586$.

| $b$ | exactly one of each digit | at most one of each digit |  |
| ---: | :--- | :--- | :--- |
| 2 | 0.5 | 1.5 |  |
| 3 | 0.25782638414217361586 | 2.60068352699931647300 |  |
| 4 | 0.12978480840622361677 | 3.60808680298641368569 |  |
| 5 | 0.061089504185883722653 | 4.56754453463005332968 |  |
| 6 | 0.027277915509357300827 | 5.49201225023269789881 |  |
| 7 | 0.011747438874515538097 | 6.38730219721336381858 |  |
| 8 | 0.0049323103515558869283 | 7.25702340207206068918 |  |
| 9 | 0.0020326818604436559280 | 8.10385281808560531093 |  |
| 10 | 0.00082589034791925293861 | 8.92994817475544342417 |  |

TABLE 1. The sum of $1 / n$ where $n$ has exactly, or at most, one of each digit, base $b$

Example 2 (a). Consider the sum of $1 / n$ where $n$ has exactly two of each decimal digit. Here, all denominators in this finite series have 20 digits. This sum can be computed to 20 decimals with
iSum $[\{0,1,2,3,4,5,6,7,8,9\},\{2,2,2,2,2,2,2,2,2,2\}, 20]$
The result is 0.000054406219429099091465 .
Example 2 (b). As part of the calculation in 2 (a), the algorithm also computed the $3^{10}$ sums of $1 / n$ where $n$ has zero, one, or two occurrences of each of the ten digits. Together, these sums comprise a finite sum that terminates after 20-digit denominators. The algorithm prints the total of these sums, that is, the sum of $1 / n$ where $n$ has at most two of every digit. This sum is about 20.58988677491808564961 .

Table 2 shows the results of similar calculations for bases 2 through 10. For base 2, the two sums are $1 / 9+1 / 10+1 / 12=53 / 180$ and $1 / 1+1 / 2+1 / 3+1 / 4+1 / 5+1 / 6+1 / 9+1 / 10+1 / 12=247 / 90$, respectively.

| $b$ | exactly two of each digit | at most two of each digit |
| ---: | :--- | :--- | :--- |
| 2 | 0.29444444444444444444 | 2.74444444444444444444 |
| 3 | 0.13728428135379622429 | 5.03830160738429130946 |
| 4 | 0.053227679167344982805 | 7.31560124709156819264 |
| 5 | 0.018595615857120543656 | 9.58154138700167499146 |
| 6 | 0.0061233385853873923485 | 11.82888191618040998185 |
| 7 | 0.0019419373425065960903 | 14.05401731726587698270 |
| 8 | 0.00060011954844577443387 | 16.25577418776059932348 |
| 9 | 0.00018199156854643633810 | 18.43419290903251052124 |
| 10 | 0.000054406219429099091465 | 20.58988677491808564961 |

TABLE 2. The sum of $1 / n$ where $n$ has exactly, or at most, two of each digit, base $b$

Example 3. Consider the sum of $1 / n$ where $n$ has exactly three of each decimal digit. This sum is about 0.0000100646100949479359866661443569 . In order to obtain this result, the algorithm also computed the $4^{10}$ sums of $1 / n$ where $n$ has zero, one, two, or three occurrences of each digit. This computation took about a week on a 2011-era PC. The sum of $1 / n$ where $n$ has at most three of each digit, also part of iSum's output, is about 34.007821434215657 463923451696316 .

Table 3 shows the results of similar calculations for bases 2 through 10 .

| $b$ | exactly three of each digit | at most three of each digit |
| ---: | :--- | :--- | :--- |
| 2 | 0.23033736086459358053 | 3.94782643525769274143 |
| 3 | 0.094468852473261028516 | 7.56978835464572925186 |
| 4 | 0.030515933974338568330 | 11.29252152684271155967 |
| 5 | 0.0088407898731766697690 | 15.06421249359298551104 |
| 6 | 0.0024115458924165521677 | 18.85575072108214171749 |
| 7 | 0.00063327051122059045079 | 22.65231884308942899556 |
| 8 | 0.00016201572624565116005 | 26.44590019584733812836 |
| 9 | 0.000040672018395991375471 | 30.23194872854812830492 |
| 10 | 0.000010064610094947935987 | 34.00782143421565746392 |

Table 3. The sum of $1 / n$ where $n$ has exactly, or at most, three of each digit, base $b$

Example 4. A. D. Wadhwa [14] considered $s_{k}=$ sum of $1 / n$ where $n$ has exactly $k 0$ 's. He showed that $s_{k}$ is strictly decreasing and that $s_{k}>19.28$ for all $k$. With iSum [0, 0], iSum [0, 1],
etc., we can calculate that

$$
\begin{aligned}
s_{0} & \approx 23.103447909420542 \\
s_{1} & \approx 23.026735341569127 \\
s_{2} & \approx 23.025860682735520 \\
s_{3} & \approx 23.025851037148538 \\
s_{10} & \approx 23.0258509299404568401819892
\end{aligned}
$$

This last result was obtained with iSum [0, 10, 25]. Notice that $s_{10}-10 \ln (10) \approx 2 \times 10^{-21}$. Question: Why is this difference so small? Answer: Bakir Farhi [4] has shown that, for any digit $d$, if $s_{k}(d)$ is the sum of $1 / n$ where $n$ has exactly $k$ occurrences of $d$, then $s_{k}(d)$ converges, from above, to $10 \ln (10)$ as $k$ approaches infinity. In base $b$, the sums approach $b \ln (b)$.

Example 5. In textbooks, most examples of convergent series have sums that are less than about 3 . Frequently-encountered subseries of the harmonic series, such as $\sum 1 / 2^{k}, \sum 1 / k^{2}$, and $\sum 1 / k$ !, come to mind. However, the examples above suggest how to construct nontrivial subseries of the harmonic series that have arbitrarily large, but computable, sums. For example, let's compute the sum of $1 / n$ where $n$ has three 0's: iSum [0, 3] gives this output:

```
sum = 23.025851037148538
    sum with exactly 0 occurrences of 0 = 23.103447909420542
    sum with exactly 1 occurrence of 0 = 23.026735341569127
    sum with exactly 2 occurrences of 0 = 23.025860682735520
    sum with exactly 3 occurrences of 0 = 23.025851037148538
    sum with at most 3 occurrences of 0 = 92.181894970873727
```

Each of the four values of "sum with exactly $k$ occurrences" is, very roughly, $10 \ln (10)$. The last line is the total of the four directly lines above it, so this total is about $4 \times 10 \ln (10)$. In general, the sum of $1 / n$ where $n$ has at most $M 0$ 's would be roughly $(M+1) \times 10 \ln (10)$. We can make this sum exceed 1000 by taking $M \geq \frac{1000}{10 \ln (10)}-1 \approx 43$.
For $M=434$, the sum exceeds 10000 ; the sum with at most 100000 's is about 10016.32364 5776401861 09739. The Kempner series considered in [11] can also have arbitrarily large sums. For example, the sum of $1 / n$ where n has no occurrence of the digit string " 314159 " is about 2302582.33386378260789202376.

Example 6. Table 4 displays, for each decimal digit $d$, the sum of $1 / n$ where $n$ has zero, one, or two occurrences of $d$. The table also shows, for example, that the sum of $1 / n$ where $n$ has at most two 9 's is about $22.92067 \ldots+23.04428 \ldots+23.02604 \ldots \approx 68.99100 \ldots$.

## 6. Further Comments

1. Suggestion for future work: the algorithm presented here allows one to sum a series whose denominators contain a chosen number of occurrences of one or more digits. For example, we

| $d$ | zero occurrences | one occurrence | two occurrences |
| :---: | :--- | :--- | :--- | :--- | :--- |
| 0 | 23.10344790942054161603 | 23.02673534156912696109 | 23.02586068273551997642 |
| 1 | 16.17696952812344426658 | 23.16401859427283204085 | 23.02727628635600571224 |
| 2 | 19.25735653280807222453 | 23.08826066275634239334 | 23.02648597376847065598 |
| 3 | 20.56987795096123037108 | 23.06741088193023010242 | 23.02627319066793505960 |
| 4 | 21.32746579959003668664 | 23.05799241338182439576 | 23.02617788539260017317 |
| 5 | 21.83460081229691816341 | 23.05272889453011749904 | 23.02612487531564760861 |
| 6 | 22.20559815955609188417 | 23.04940997329550055704 | 23.02609154986488712587 |
| 7 | 22.49347531170594539818 | 23.04714619019864185083 | 23.02606886491441507436 |
| 8 | 22.72636540267937060283 | 23.04551390798215553342 | 23.02605253084569367648 |
| 9 | 22.92067661926415034816 | 23.04428708074784831968 | 23.02604026596124378845 |

TABLE 4. The sum of $1 / n$ where $n$ has exactly one, two, or three of each digit, $d$
can compute the sum of $1 / n$ where $n$ has exactly one 3 , one 1 , and one 4 . (The sum is about 1.69447989917979092497 ). But how do we compute the sum of $1 / n$ where $n$ has, say, exactly one occurrence of a multi-digit string such as " 314 "?

For Kempners' series, the algorithm in [1] allows one to sum a series when one digit is excluded from the denominators. Like the algorithm presented here, that algorithm is based on the way that $i$-digit denominators in the series gave rise to the $(i+1)$-digit denominators.

Schmelzer [11] generalized the algorithm in [1] to handle the case when a multi-digit string of digits is excluded from the denominators. For example, the sum of $1 / n$ where $n$ has no occurrence of " 314 " is about 2299.8297827675 18338. This latter algorithm works by partitioning the $i$-digit denominators into several subsets, then extrapolating to $(i+1)$-digit denominators on each set separately. Can the algorithm presented here be generalized in a similar way to compute the sum of $1 / n$ where $n$ has one occurrence of " 314 "? The author has not pursued the matter. The ability to sum these series might lead to interesting results.
2. The functions iPartialSum and iPartialSumThreshold (briefly mentioned in Section 3) allow us to estimate how large denominators must be in order for the partial sum to exceed a given threshold.

For example, consider the $s_{10}$ series in Example 4 above, which begins

$$
\frac{1}{10000000000}+\frac{1}{20000000000}+\frac{1}{30000000000}+\ldots
$$

The following calculations with iPartialSum show that it is not until we include some denominators between $10^{62}$ and $10^{63}$ that the partial sums of the $s_{10}$ series exceed 1 :

```
iPartialSum[0, 10, 62] = 0.994418222777579
iPartialSum[0, 10, 63] = 1.099295133607324 .
```

The first result above shows that the sum of the series including denominators $<10^{62}$ is about 0.99442 . The sum of the series including denominators $<10^{63}$ is about 1.09930.

The function iPartialSumThreshold eliminates the need to obtain the above " 62 " and " 63 " by trial and error. iPartialSumThreshold[0, 10, 1.0] gives

$$
\{62,0.9944182227775792,63,1.09929513360732359\} .
$$

Interestingly, the sum of the $s_{10}$ series is greater than the sum of Kempner's "no 9" series, even though the "no 9" series begins with much larger terms

$$
\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{8}+\frac{1}{10}+\ldots
$$

It takes a long time for the partial sum of the $s_{10}$ series to exceed the sum of the "no 9 " series. The functions iPartialSum and iPartialSumThreshold show that it is not until we include denominators having 210 digits that the partial sum of the $s_{10}$ series exceeds $22.92067 \ldots$, the sum of the "no 9" series:
iPartialSumThreshold[0, 10, 22.92067] gives this result:
$\{209,22.917796696018994,210,22.924073628793615\}$.
3. Here is an even more striking example. In the sum of $1 / n$ where $n$ has exactly 100 zeros, the first term is $1 /$ googol:

$$
s_{100}=\frac{1}{10^{100}}+\frac{1}{2 \cdot 10^{100}}+\frac{1}{3 \cdot 10^{100}}+\ldots
$$

yet the sum converges very slowly to about $10 \ln (10)+1.00745721706770421142 \cdot 10^{-197} \approx$ $23.02585 \ldots$. The following calculation shows that, to make the partial sum exceed $10 \ln (10)$, we need at least some denominators $>10^{7225}$ :
$\mathrm{a}=\mathrm{iPartialSumThreshold}[0,100,10 * \log [10], 220]$;
The result is, roughly, \{7225, 23.02585..., 7226, 23.02585...\}. Comparing the second and fourth elements of this list to $10 \ln (10)$, we get,

```
a[[2]] - 10 Log[10] = -2.83888407016547054930*10^-200
a[[4]] - 10 Log[10] = 8.544994808119088314378*10^-199
```

Note that we must do this calculation to over 200 decimal places in order to observe that the partial sum really exceeds $10 \ln (10)$.

Table 5 shows partial sums of this series at various thresholds. In the table, $t_{1}$ and $t_{2}$ are the partial sums for $n<10^{m}$ and $n<10^{m+1}$.

For each $d$, table 6 shows the sum of $1 / n$ where $n$ has 100 occurrences of the digit $d$. These were computed using iSum [d, 100, 250].

By contrast, for base 2 , the sum of $1 / n$ where $n$ has exactly 100 (binary) zeros is about $2 \ln 2+$ $4.5853670842 \cdot 10^{-49}$. This sum is calculated using iSum [0, 100, 80, 2]. Similarly, the sum of $1 / n$ where $n$ has exactly 100 (binary) ones is about $2 \ln 2+2.0516487683 \cdot 10^{-48}$.

| Threshold | $m$ | $t_{1}$ | $t_{2}$ |
| :--- | ---: | :--- | :--- |
| 1 | 852 | .99153 | 1.01670 |
| 2 | 882 | 1.98058 | 2.02260 |
| 10 | 990 | 9.95559 | 10.05188 |
| 20 | 1116 | 19.96750 | 20.01555 |
| 22.92067 | 1276 | 22.92027 | 22.92298 |
| 23 | 1327 | 22.99988 | 23.00062 |
| 23.02 | 1376 | 23.01996 | 23.02014 |
| 23.025 | 1434 | 23.02498 | 23.02501 |
| 23.0258 | 1512 | 23.02579997 | 23.02580187 |
| 23.02585 | 1611 | 23.025849977 | 23.025850017 |
| 23.0258509299 | 1829 | 23.0258509298996 | 23.0258509299016 |
| $10 \ln 10$ | 7225 | $10 \ln 10-2.84 \cdot 10^{-200}$ | $10 \ln 10+8.54 \cdot 10^{-199}$ |
|  | TABLE 5. Various thresholds for $s_{100}$ |  |  |


5. Now consider $s_{200}$. The first term in $s_{200}$ is $1 /$ googol $^{2}$. Even though the series begins with these very small terms,

$$
s_{200}=\frac{1}{10^{200}}+\frac{1}{2 \cdot 10^{200}}+\frac{1}{3 \cdot 10^{200}}+\ldots
$$

the sum of this series is still about $10 \ln (10)+1.2562827028880148800327775 \cdot 10^{-393}$. This sum was computed using iSum [0, 200, 420], and took about 2 hours on a 2011-era PC.
6. Likewise, the first term in $s_{\text {googol }}$ is the tiny $1 /$ googolplex; nevertheless, thanks to Farhi's result [4], we know that the sum of the series must (very slightly) exceed $10 \ln (10)$.
7. Are the sums we consider here rational or irrational? The series in Examples 1-3 have a finite number of terms, and are therefore rational. But there is no particular reason to think the other sums are rational. And, because there are more irrationals than rationals, any bets should be placed on the sums being irrational.

However, we do know this: a theorem of Borwein [3, Theorem 1] states that if $b$ is an integer such that $|b|>1$ and $c$ is a non-zero rational number, then

$$
\sum_{n=1}^{\infty} \frac{1}{b^{n}+c}
$$

is irrational (assuming that $b^{n}+c$ is not zero for any $n \geq 1$ ). If we take $b=10$ and $c=-1$, this theorem says that

$$
\sum_{n=1}^{\infty} \frac{1}{10^{n}-1}=\frac{1}{9}+\frac{1}{99}+\frac{1}{999}+\ldots
$$

is irrational. Multiply this series by $9 / d$ and we see that if the denominators of a series consist of any single non-zero digit $d$, then the sum is irrational. The same is true in other bases $b$.

Let $x_{d}$ be the sum of $1 / n$ where $n$ has one occurrence of $d$. For each digit $d=0,1, \ldots, 9$, the author computed $x_{d}$ to 5010 decimal places. For none of these sums did the Mathematica function Rationalize find a rational number $p / q$ such that

$$
\left|x_{d}-\frac{p}{q}\right|<\frac{10^{-4}}{q^{2}}
$$

For $d=0$ and $d=9$, the author also computed the corresponding sums to 10010 decimal places. Rationalize also could not find a $p / q$ meeting the above criterion for either of these sums.

Mathematica also has a function, RootApproximant, which takes as input a number $c$ and attempts to find a polynomial with integer coefficients that has $c$ as a root. If $c$ is a number known to $n$ digits, then one can usually find such a polynomial for which the total number of digits in the coefficients is about $n$, even if $c$ is not algebraic. However, if the total number of digits in the coefficients is much less than $n$, this strongly suggests that $c$ is, indeed, algebraic, with $c$ being a root of that polynomial. All ten 5010 -decimal place approximations $x_{d}$ with $d=0,1, \ldots, 9$ were tested with RootApproximant. In no case did RootApproximant find a polynomial of degree $\leq 10$ where the total number of digits in the coefficients was less than 5000. Likewise, for the two 10010 decimal place approximations with $d=0$ and $d=9$, RootApproximant could not find polynomials of degree $\leq 10$ where the total number of digits in the coefficients was less than 10000.

## References

[1] Robert Baillie, Sums of reciprocals of integers missing a given digit, American Mathematical Monthly $\mathbf{8 6}$ (May, 1979), pp. 372-374.
Available through JSTOR at http://dx.doi.org/10.2307/2321096 .
[2] Ralph P. Boas, Jr. and John W. Wrench, Jr., Partial Sums of the Harmonic Series, American Mathematical Monthly 78 (October, 1971), pp. 864-870.
Available through JSTOR at http://dx.doi.org/10.2307/2316476.
[3] Peter Borwein, On the Irrationality of Certain Series, Math. Proc. Cambridge Philos. Soc. 112 (1992), 141-146.
Available for U.S. \$45.00 at http://dx.doi.org/10.1017/S030500410007081X ;
available free at http://cgi.cecm.sfu.ca/~pborwein/PAPERS/P59.pdf .
[4] Bakir Farhi, A Curious Result Related to Kempner's Series, American Mathematical Monthly 115 (December, 2008), pp. 933-938.
Available through JSTOR at http://www.jstor.org/stable/27642640;
a preprint is online at http://arXiv.org/pdf/0807.3518.pdf.
[5] T. Foregger, Helping Professor Umbugio. Solution to Problem E2533, American Mathematical Monthly 83 (August/September, 1976), pp. 570-571.
[6] G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Oxford, 4th edition, 1960.
[7] Julian Havil, Gamma: Exploring Euler's Constant, Princeton University Press, Princeton, 2003.
[8] Frank Irwin, A Curious Convergent Series, American Mathematical Monthly, 23 (May, 1916), pp. 149-152. Available free through JSTOR at http://dx.doi.org/10.2307/2974352.
[9] A. J. Kempner, A Curious Convergent Series, American Mathematical Monthly, 21 (February, 1914), pp. 48-50.
Available free through JSTOR at http://dx.doi.org/10.2307/2972074.
[10] E. S. Pondiczery, Elementary Problem E2533, American Mathematical Monthly 82 (April, 1975), p. 401.
[11] Thomas Schmelzer and Robert Baillie, Summing a Curious, Slowly Convergent Series, American Mathematical Monthly 115 (June/July, 2008), pp. 525-540.
Available through JSTOR at http://www.jstor.org/stable/27642532;
Mathematica code that implements the algorithm in the Schmelzer/Baillie article is at http://library.wolfram.com/infocenter/MathSource/7166.
[12] Sequence A004080, The On-Line Encyclopedia of Integer Sequences, OEIS Foundation, http://oeis.org/A004080.
[13] Sequence A082912, The On-Line Encyclopedia of Integer Sequences, OEIS Foundation, http://oeis.org/A082912.
[14] A. D. Wadhwa, Some Convergent Subseries of the Harmonic Series, American Mathematical Monthly 85 (October, 1978), pp. 661-663.
Available through JSTOR at http://dx.doi.org/10.2307/2320338
[15] Eric Weisstein, Digit, MathWorld-A Wolfram Web Resource, http://mathworld.wolfram.com/Digit.html.
[16] Eric Weisstein, Negative Binomial Series, MathWorld-A Wolfram Web Resource, http://mathworld.wolfram.com/NegativeBinomialSeries.html.
[17] David Wells, The Penguin Dictionary of Curious and Interesting Numbers, Penguin Books, London, 1997.
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## 7. The Mathematica Code for Irwin Sums

```
(* :Title: irwinSums.m -- a package to compute sums of irwin series *)
(* :Mathematica Version: 4.0, 5.2, 6.0, 7.0, 8.0, and 9.0 *)
(* :Package Version: 1.5, August 18, 2013:
    (1) Changes to make the output more self-explanatory.
    (2) Fixed output function nf[ ] so for numbers between 0 and 1,
                we display exactly the requested number of decimal places
                instead of one more than the requested number.
    (3) In the NSum[ ] calls in computeMaxPowerNeeded[ ], added
                AccuracyGoal->nDecimals + 10
                to eliminate the "NIntegrate::izero ..." warning when running
                iSum[0, 200, 420]. This prevents local variable ns from being set to 0.
    Version 1.4, December 21, 2008:
        Cosmetic changes to comments.
    Version 1.3:
        added iPartialSumThreshold[ ] to compute how many digits are required
        in the denominators for the partial sum to reach a given threshold.
    Version 1.2:
        added setPrintLevel[ ] user function to control the amount of displayed output.
                default value = 1. user can set the value to 0 to suppress all output,
            including "sum for k occurrences ...".
*)
(* :Author: Robert Baillie *)
(* :Copyright: none. This code is in the public domain.
    Users are encouraged to improve this code.
*)
(* :Summary:
    This package computes sums of "Irwin" series.
    The harmonic series 1/1 + 1/2 + 1/3 + ... + 1/n + ... diverges.
    This means that the sum can be made as large as desired by adding
    enough terms.
    Suppose we delete from this series all terms with a "9" in the
    denominator. That is, we delete 1/9, 1/19, 1/29, etc. Then the
    remaining series
        1/1 + 1/2 + . . + 1/8 + 1/10 + ... + 1/18 + 1/20 + ...
    converges to a sum less than 80. This was proved by A. J. Kempner
    in 1914. The sum of this series is about 22.92067 66192 64150 .
    In 1916, Irwin proved that the sum of 1/n where n has at most a fixed
```

```
    number of occurrences of one or more digits, is a convergent series.
    For example, this Irwin series, where the denominators have exactly one 9,
        1/9 + 1/19 + 1/29 + ...
    converges. The sum of this series is about 23.04428 70807 47848 .
    At first glance, it appears that this is just the harmonic series
    with 90% of the terms removed, and would therefore diverge just like
    the harmonic series does. However, this series thins out much
    more than than the harmonic series. For example, the vast majority of
    100-digit integers have more than one 9, so would be excluded from the sum.
    Irwin's result is a generalization of Kempner's. In Kempner's series,
    the digit 9 occurs zero times. It also follows that the sum of 1/n
    where n has exactly n1 occurrences of digit d1, n2 occurrences of d2,
    etc., converges.
    These series converge much too slowly to get even a rough
    estimate of the sums by simply adding up the terms.
    This package computes the sums of Irwins' series to high precision.
*)
(* :Keywords: kempner, irwin, harmonic series *)
(* :Sources:
    The original article that proves convergence of these series is:
        Frank Irwin, "A Curious Convergent Series",
        American Mathematical Monthly, vol. 23, pages 149-152 (May, 1916).
        Available free through JSTOR at http://dx.doi.org/10.2307/2974352 .
    See also:
        A. J. Kempner, "A Curious Convergent Series",
        American Mathematical Monthly, vol. 21, pages 48-50 (February, 1914).
        Available free through JSTOR at http://dx.doi.org/10.2307/2972074 .
        Robert Baillie, "Summing The Curious Series of Kempner and Irwin",
        May, 2008. Available at http://arxiv.org/abs/0806.4410 .
*)
BeginPackage[ "irwinSums`" ]
(* :Examples:
    iSum[9, 0] = sum of 1/n where n has no 9's = 22.920676619264150 .
    iSum[9, 1] = sum of 1/n where n has exactly one 9 = 23.044287080747848.
    iSum[{9, 3}, {2, 0}] = sum of 1/n where n has two 9's and no 3's
            =2.593253652747189.
```

iSum [9, 2] (this computes the sum of $1 / \mathrm{n}$ where n has exactly two 9 's)
Detailed explanation of the output:
Line
1 sum $=23.026040265961244$
2 sum with exactly 0 occurrences of $9=22.920676619264150$
3 sum with exactly 1 occurrence of $9=23.044287080747848$
4 sum with exactly 2 occurrences of $9=23.026040265961244$
5 sum with at most 2 occurrences of $9=68.991003965973242$
Line 1: This is the requested sum of $1 / n$ where $n$ has exactly two 9 's.
Lines 2 - 4: Three sums were computed as intermediate steps in this calculation:
the sums of $1 / \mathrm{n}$ where n has exactly zero, one, or two 9 's.
Line 5 shows the total of the three sums in lines 2-4. This is just the sum of $1 / n$ where $n$ has at most two 9 's.
iSum [\{9, 3\}, \{2, 1\}] (compute the sum of $1 / \mathrm{n}$ where n has exactly two 9 's and one 3 ) Detailed explanation of the output:
Line
1 sum $=4.169026439566082$
2 sum with at most 29 's and at most $13=34.282119242240692$
Line 1: This is the requested sum of $1 / n$ where $n$ has exactly two 9 's and one 3 .
Line 2: Six sums were computed as intermediate steps in this calculation. They are
the sums of $1 / n$ where $n$ has exactly:
zero 9's and zero 3's
zero 9's and one 3
one 9 and zero 3 's
one 9 and one 3
two 9's and zero 3 's
two 9's and one 3.
Line 2 shows the total of these six sums. This is just the sum of $1 / n$ where n has at most two 9 's and one 3 .
*)

```
(* usage messages for this context, and for the exported functions *)
irwinSums::usage = "irwinSums.m is a package that computes sums of Irwin series.\r\n
This package has functions iSum, iSumFormatted, iPartialSum, iPartialSumThreshold,\
    and setPrintLevel.\r\n
    iSum[d, k]: computes the sum of 1/n where n has exactly k occurrences of the digit d.\r\n
    iSum[{d1, d2 ,\ldots.}, {k1, k2, ...}]: sum of 1/n where n has exactly k1 occurrences\
    of d1, k2 occurrences of d2, etc.\r\n
    iSum[..., ..., m] rounds the result to m decimals.\r\n
    iSum[..., ..., m, b] does calculations with digits in base b.\r\n
    iSumFormatted[...] is the same as iSum, but formats the output in groups of 5 digits.\r\n
    iPartialSum[d, k, p] or iPartialSum[{d1, d2 ,...}, {k1, k2, ...}, p] gives the partial sum
    through denominators < 10^p.\r\n
    iPartialSumThreshold[d, k, s] and iPartialSumThreshold[{d1, d2 ,...}, {k1, k2, ...}, s] tell
    how many digits are needed in the denominators in order to make the partial sum\
    reach your threshold, s.\r\n
    setPrintLevel sets the print level to control the amount of output." ;
```

```
iSum::usage = "iSum[d, k] computes the sum of the series 1/n where the digit d occurs\
    exactly k times in n.\r\n
    iSum[{d1, d2, ...}, {k1, k2, ...}] is the sum where the digit d1 occurs exactly k1 times, d2\
    occurs exactly k2 times, etc.\r\n
    iSum[d, k, m] and iSum[{d1, d2, ...}, {k1, k2, ...}, m] give the sum to m decimal places.\r\n
    All results are rounded to the number of decimal places shown.\r\n
    Examples:
        iSum[9, 0] = sum of 1/n where n has no 9's = 22.920676619264150 .\r\n
        iSum[9, 1] = sum of 1/n where n has exactly one 9 = 23.044287080747848.\r\n
    You can also specify a list of conditions:
        iSum[{9, 3}, {2, 0}] = sum of 1/n where n has two 9's and no 3's = 2.593253652747189 . \r\n
    iSum takes an optional third parameter: the number of decimal places. If not specified,
    the default is 15 decimals.\r\n
    iSum takes an optional fourth parameter: the base. If not specified, base 10 is used.
    If you specify the fourth parameter (base), you cannot omit the third parameter\
    (the number of decimals).
    Examples of calculations in base 2:
        iSum[1, 1, 20, 2] = sum of 1/n where n has one 1 in base 2.\
    The sum is 1/1 + 1/2 + 1/4 + .. = 2 .
        iSum[1, 3, 20, 2] = sum of 1/n where n has three 1's in base 2 = 1.42859154585263812400 .
        iSum[0, 0, 20, 2] = sum of 1/n where n has no 0's in base 2 = 1.60669515241529176378 ." ;
iSumFormatted::usage = "Same as iSum, but iSumFormatted formats the output\
    into groups of 5 digits." ;
iPartialSum::usage = "iPartialSum[d, k, p] computes the partial sum of 1/n\
    where n has k occurrences of d, for n < 10^p.
    Similarly for iPartialSum[{d1, d2 ,...}, {k1, k2, ...}, p].
    Example:
    iSum[9, 0] = 22.920676619264150; iPartialSum[9,0, 30] = 21.971055078178619.
    The optional 4th parameter iPartialSum[..., ..., p, m] gives the partial sum to m decimals.
    The optional 5th parameter is the base:
        iPartialSum[1, 1, 6, 15, 2] = partial sum of 1/n where n has one 1 in base 2, n < 2^6,\
    to 15 decimals.
        Note: this is just 1/1 + 1/2 + 1/4 + 1/8 + 1/16 + 1/32 = 63/32 = 1.96875." ;
```

iPartialSumThreshold::usage = "iPartialSumThreshold[d, k, s] tells how many digits are needed $\backslash$
in the denominators in order to make the partial sum equal to or greater than your threshold, s.
Four numbers are returned: \{ d1, d2, s1, s2 \}. d 2 is the number of digits required in the $\backslash$
denominators to make the partial sum $>=s . d 1=d 2-1 . \quad s 1$ and $s 2$ are the partial sums $\backslash$
for d 1 and d 2 .
Given $s$, the output sums $s 1$ and $s 2$ will normally be such that $s 1<s<=s 2 . \backslash r \backslash n$
Example 1. iPartialSumThreshold[9, 1, 23] computes about how far we need to go\}
to reach a partial sum of 23.
The output is $\{80,22.995762680948152,81,23.000125707332644\}$.
Therefore, denominators of 81 digits are required. The sum through $\mathrm{n}<10^{\wedge} 80$ is about $22.99576 . \$

```
The sum through n < 10^81 is about 23.00012.
iPartialSum[ ] confirms this result:
    iPartialSum[9, 1, 80] = 22.995762680948152
    iPartialSum[9, 1, 81] = 23.000125707332644 .\r\n
Example 2. This shows what could go wrong if your threshold is a floating-point number.
    iPartialSumThreshold[9, 1, 23.044287080747] returns {-1, -1, -1, -1} because the
    threshold is too close to the sum of the entire series. The error message tells you
    to specify more accuracy. You can do this with Mathematica's backquote notation:
    23.044287080747``25 = 23.044287080747000000000000.
        iPartialSumThreshold[9, 1, 23.044287080747``25]
    returns the correct result
        {327, 23.04428708074693636344610077, 328, 23.04428708074702511802366170}
    You can also use double quotes to convert the threshold to a string:
        iPartialSumThreshold[9, 1, \"23.044287080747\"]
    also returns the correct result
        {327, 23.044287080746936363, 328, 23.044287080747025118} .\r\n
Other forms: iPartialSumThreshold[{d1, d2 ,...}, {k1, k2, ...}, s]. You can also specify\
the number of decimals and the base: iPartialSumThreshold[d, k, s, nDecimals, base] and\
iPartialSumThreshold[{d1, d2 ,...}, {k1, k2, ...}, s, nDecimals, base]." ;
setPrintLevel::usage = "Set the print level = 0, 1, 2, 3, 4. Larger values produce more output. \
Default = 1." ;
getPrintLevel::usage = "Display the current print level." ;
Begin["`Private`"] (* begin the private context (implementation) *)
Off[ General::spell1 ];
Off[ General::spell ];
(* here are the global variables *)
(* sjk and sjkPrev are tables (arrays) of size maxJ by maxIndexUsed. the
    first dimension is a power, the second is an index number 1 .. maxIndexUsed.
*)
sjk;
sjkPrev;
cumulativeSums1; (* this is a small array, size = countList[[1]] + 1 *)
iSumPrintLevel = 1; (* this is the default print level *)
Unprotect[iSum, iSumFormatted, iPartialSum, iPartialSumThreshold, setPrintLevel, getPrintLevel];
getDigitsforNFunction[y_, nDec_] :=
```

```
Module[
(* this computes the number of digits to send to the N[ ] function.
    we want to display an extended-precision number to (nDec) decimal places,
    not counting the leading zeros if the number is between 0 and 1.
    if Abs[y] < 1, then N[y, nDec] displays the correct number of decimal places.
    otherwise, in N[y, nDec], the number of digits shown includes digits to the left of
    the decimal point. in this case, we must increase the number of digits sent to N[].
*)
    { nExtra },
    If [Abs[y] >= 1,
        nExtra = 1 + Floor[ Log[10, Abs[y]] ],
        nExtra = 0
    ];
    nExtra + nDec
] (* end of getDigitsforNFunction *)
(* nf0 formats a number to (nDec) decimal places *)
nf0[y_, nDec_] := If[y == 0, O, N[y, getDigitsforNFunction[y, nDec]] ];
(* nf1 formats a number to (nDec) decimal places, with a space every 5 digits *)
nf1[y_, nDec_] := If [y == 0, 0,
    NumberForm[y, getDigitsforNFunction[y, nDec], DigitBlock->5, NumberSeparator->{""," "}] ];
(* nf[ ] does number formatting. show (nDec) decimal places, not counting leading zeros
    if the number is between 0 and 1.
    if the iFlag is 1, this does "NumberForm" formatting with spaces separating groups of
    5 digits; otherwise, the standard N[ ] function is used.
*)
nf[y_, nDec_, iFlag_:0] := If [iFlag == 0, nf0[y, nDec], nf1[y, nDec] ];
(* power2 is just like Power[], but power2 returns 0^0 = 1 without warning messages *)
power[0, 0] := 1; (* special case *)
power[x_, y_] := Power[x, y] ; (* in other cases, same as the built-in function Power[] *)
isThisASpecialDigit[iDigit_, nConditions_, digitList_List] :=
Module[
    (* if iDigit is in the list of special digits,
        then return the 1-based location in the list;
        otherwise, return 0. *)
    { j },
    For[j = 1, j <= nConditions, j++,
        If[iDigit == digitList[[j]], Return[j]]
    ];
    0 (* return this value *)
```

```
]; (* end of isThisASpecialDigit *)
```

```
bn[iBase_Integer, n_Integer, nConditions_Integer, digitList_?VectorQ] :=
Module[
    (* this is for a set of special digits
        compute sum(k = 0 through iBase-1) of k^n (for k not a special digit).
        example: if iBase = 10, one special digit (d),
        compute bn = 9 (n = 0), bn = 0^n + 1^n + ... + 9^n - d^n (n > 0).
    *)
    { bn = 0, k },
    If [n == 0, Return[iBase - nConditions] ]; (* iBase minus number of special digits *)
    For[k = 1, k <= iBase - 1, k++,
        If[isThisASpecialDigit[k, nConditions, digitList] != 0, Continue[] ];
        bn += power[k, n];
    ];
    Return[bn];
]; (* end of bn *)
getArrayIndexFromList[n_Integer, ci_?VectorQ, iList_?VectorQ] :=
Module[
    (* this is called with nConditions = n and countList = ci.
        this function is the inverse of getListFromArrayIndex. *)
    { i, iSum = 0, iProd = 1 },
    iSum = iList[[1]];
    For[i = 2, i <= n, i++,
        iProd = iProd * (ci[[i-1]] + 1);
        iSum += iList[[i]] * iProd;
    ];
    iSum + 1 (* return this value; all arrays in mathematica are 1-based *)
]; (* end of getArrayIndexFromList *)
getListFromArrayIndex[iArrayIndex_Integer, n_Integer, ci_?VectorQ] :=
Module[
(*
    given an array index, find the list iList[1..n] that gives this index.
    this is called with nConditions = n and countList = ci.
    this function is the inverse of getArrayIndexFromList.
*)
    { i, iProd = 1, iTemp, iList = Table[0, {n}] },
    iProd = Product[ci[[i]] + 1, {i, n-1}];
```

```
    iTemp = iArrayIndex;
    iTemp = iTemp - 1; (* for mma *)
    For[i = n-1, i >= 1, i--,
        iList[[i+1]] = Floor[ iTemp/iProd ];
        iTemp = iTemp - iProd*iList[[i+1]]; (* remainder *)
        iProd = Floor[ iProd/(ci[[i]] + 1) ];
    ];
    iList[[1]] = iTemp;
    iList (* return this list *)
]; (* end of getListFromArrayIndex *)
updateCumulativeSums1[iBase_Integer, nConditions_Integer, countList_?VectorQ] :=
Module[
    (* if there is one condition, and we are summing the series for k occurrences
        of a digit, then keep track of all cumulative sums for i = 0, 1, 2, ..., k
        occurrences of that digit. *)
    { i, iArrayIndex, kOccurFound },
    kOccurFound = Table[ 0, {iBase + 1} ];
    For[i = 0, i <= countList[[1]], i++,
        kOccurFound[[1]] = i;
        iArrayIndex = getArrayIndexFromList[nConditions, countList, kOccurFound];
        cumulativeSums1[[i+1]] += sjkPrev[[ 1, iArrayIndex ]];
    ];
]; (* end of updateCumulativeSums1 *)
printSumsWithKOccurrences[nDec_Integer, nConditions_Integer, digitList_?VectorQ,
                countList_?VectorQ, iFormatted_Integer] :=
Module[
    (* if there is just one condition, then print
        the sum for exactly k occurrences, k = 0, 1, 2, ... countList[[1]]. *)
    { k, occurStr },
    If[nConditions != 1, Return[] ];
    For[k = 0, k <= countList[[1]], k++,
        If[k == 1, occurStr = "occurrence ", occurStr = "occurrences"];
        Print[" sum with exactly ", k, " ", occurStr, " of ", digitList[[1]], " = ",
            nf[cumulativeSums1[[k+1]] , nDec, iFormatted] ]
    ]; (* end for k loop *)
]; (* end of printSumsWithKOccurrences *)
```

```
printAtMostSumNCond[nConditions_Integer, digitList_?VectorQ, countList_?VectorQ, sumSmallerK_,
                nDec_Integer, iFormatted_Integer] :=
Module[
    (* print the "at most" sums on the digits when there are at least two conditions;
        nConditions equals the length of the digitList and countList vectors.
        for two conditions, show both conditions; for 3 or more, show the first and the last
        for iSum[{7, 8}, {2, 3}], print
            sum with at most 2 7's and at most 3 8's = 54.974509900258921
        iSum[{7, 8, 9}, { 2, 3, 4}]
            sum with at most 2 7's, ..., and at most 4 9's = 51.551633838456648
    *)
    { k, dotStr, s1Str, s2Str },
    If[nConditions < 2, Return[] ];
    If[nConditions == 2, dotStr = "", dotStr = ", ...," ];
    If[countList[[1]] == 1, s1Str = "", s1Str = "'s" ];
    If[countList[[nConditions]] == 1, s2Str = "", s2Str = "'s" ];
    Print[" sum with at most ", countList[[1]], " ", digitList[[1]],
                s1Str, dotStr," and at most ", countList[[nConditions]], " ",
            digitList[[nConditions]], s2Str, " = ", nf[sumSmallerK, nDec, iFormatted]
    ];
]; (* end of printAtMostSumNCond *)
directSummation[iBase_Integer, numDigits_Integer, maxPower_Integer, nConditions_Integer,
                digitList_?VectorQ, countList_?VectorQ, nDec_Integer] :=
Module[
    (* compute partial sums by directly adding terms whose denominators have (numDigits) digits *)
    { nTerms, (* return this value *)
        iStart, iLast, i, iNumber, iDigitPos, iQuot, iRemainder, iDigit, jPower,
        k, kFound, iOK, iMatch, sumK = 0,
        kOccurFound,
        iArrayIndex,
        xNumb, xRecip, xRecipPower
    },
    kOccurFound = Table[0, {nConditions}]; (* 1-based, one for each special digit *)
    nTerms = 0;
    iStart = iBase^(numDigits - 1);
    iLast = iBase^numDigits - 1;
    For[i = iStart, i <= iLast, i++,
        (* break up (i) into individual digits *)
        (* clear all counts *)
        For[k = 1, k <= nConditions, k++, kOccurFound[[k]] = 0];
        iNumber = i;
        For[iDigitPos = 1, iDigitPos <= numDigits, iDigitPos++,
            iQuot = Floor[ iNumber/iBase ];
```

```
    iRemainder = iNumber - iBase*iQuot;
    iDigit = iRemainder;
    kFound = isThisASpecialDigit[iDigit, nConditions, digitList];
    (* if this is the digit for condition number 2, increment the 2nd total *)
    If[kFound > 0,
        kOccurFound[[kFound]]++ ;
    ];
    iNumber = iQuot;
]; (* end for iDigitPos loop *)
iOK = 1;
iMatch = 1;
For [k = 1, k <= nConditions, k++,
    If[kOccurFound[[k]] > countList[[k]],
        iOK = 0; (* this (i) had too many occurrences of one or more digits *)
        Goto[nextI]; (* go to next iteration of main (i) loop *)
    ];
    If[kOccurFound[[k]] < countList[[k]], iMatch = 0];
]; (* end for k loop *)
If[iMatch == 1, (* (i) has n1 occurrences of digit d1, n2 of d2, etc. *)
    nTerms = nTerms + 1
];
xNumb = i;
xRecip = N[1/xNumb, nDec];
(* now compute the place in the sjkPrev array where this sum belongs *)
iArrayIndex = getArrayIndexFromList[nConditions, countList, kOccurFound];
sjkPrev[[ 1, iArrayIndex ]] += xRecip;
xRecipPower = xRecip; (* xNumb^(-1) *)
For[jPower = 2, jPower <= maxPower, jPower++,
    xRecipPower = xRecipPower*xRecip; (* xNumb^(-jPower) *)
    sjkPrev[[ jPower, iArrayIndex ]] += xRecipPower;
];
Label[nextI];
    ]; (* end For i loop *)
    nTerms (* return this value *)
]; (* end of directSummation *)
computeMaxPowerNeeded[iBase_Integer, nDecimals_Integer, dsDigits_Integer] :=
Module[
    (* enter with nDecimals, which specifies the desired tolerance,
        and dsDigits = 2nd parameter to directSummation.
        let a = iBase^(dsDigits - 1), b = iBase^dsDigits - 1, and
```

```
    eps = 10^-nDecimals.
    directSummation computes Sum[1/n^j, {n, a, b}]
    for j = 1, 2, ..., k.
    we want to estimate the value of k that makes
        Sum[1/n^k, {n, a, b}] < eps .
        in bases 2 through 10, a good approximation is
            Log[iBase, 10] * nDecimals/(dsDigits - 1).
        if this is not close to a value that makes the sum < eps, then
        we must solve an equation. we estimate the sum by an integral:
            Integrate[x^-c, {x, a, b}] = (b^(1-c) - a^(1-c))/(1-c).
        then we find the value of c that makes this < eps.
    *)
    { eps, a, b, r0, k, ns, c, r },
    eps = 10^(-nDecimals); (* maximum tolerance *)
    (* a and b are the lower and upper limits in directSummation *)
    a = iBase^(dsDigits - 1);
    b = iBase^dsDigits - 1;
    (* this r0 is usually a good approximation to the value we want *)
    r0 = Ceiling[ Log[iBase, 10] * nDecimals/(dsDigits - 1) ];
    (* find a value that guarantees NSum < eps *)
    For[k = r0, k <= 10*r0, k++,
(* ns = NSum[1/n^k, {n, a, b}]; *)
    ns = NSum[1/n^k, {n, a, b}, AccuracyGoal->nDecimals + 10];
    ns = Re[ns]; (* needed only for mathematica version 5.2 *)
    If[ns < eps, Goto[loopDone] ];
    ]; (* end for k loop *)
    (* if we get here, we will have to solve the equation *)
    r = FindRoot[(b^ (1 - c) - a^ (1 - c) )/(1 - c) == eps, {c, r0} ];
    r0 = Ceiling[c /. r]; (* get the numerical value from the rule *)
    (* find a value that guarantees NSum < eps *)
    For[k = r0, k <= 10*r0, k++,
(* ns = NSum[1/n^k, {n, a, b}]; *)
        ns = NSum[1/n^k, {n, a, b}, AccuracyGoal->nDecimals + 10];
        ns = Re[ns]; (* needed only for mathematica version 5.2 *)
        If[ns < eps, Goto[loopDone] ];
    ]; (* end for k loop *)
    Return[ -r0 ]; (* return this negative value to indicate an error *)
Label[loopDone];
    r0 + 2 (* add 2 more powers just to be safe *)
]; (* end of computeMaxPowerNeeded *)
computeIrwinSum[iBase_Integer, digitList_?VectorQ, countList_?VectorQ,
                nDecimals0_Integer, iFormatted_Integer,
                nDigits_:0, threshold_:-1 ] :=
```

```
Module[
    (* this private function is the main calculation routine for the package.
        if nDigits > 0, this computes the partial sum through denominators
        of (nDigits) digits, that is, denominators that are < 10^nDigits,
        and then terminates without computing the complete sum. this parameter
        is used when the user calls iPartialSum[ ].
        threshold is used when the user calls iPartialSumThreshold.
        this usually returns a single number: either the sum >= 0, or an error value < 0.
        however, if a threshold > 0 was specified as input, then a 2-element list
        { sum, digit number } is returned.
    *)
    {
        (* iPrint
            = O for no output except for the final result,
            = 1 for minimal output,
            = 2 for some extra output,
            = 3 for some output for each iteration of the loop,
            = 4 for more output during the loop
    *)
    iPrint,
    nShow = 10, (* number of digits, for displaying during the loop *)
    nConditions, (* number of digits with a condition placed on them *)
    directSumDigits,
    nDecimals, nDec,
    i, j, nMax, n, j1, k1, (* index values in various loops *)
    iDigit, jPower, iDigitStart, k, numTerms, iAllTiny,
    (* given a digit distribution, the array index values tell
        where to store a sum in the sjk and sjkPrev arrays *)
    iArrayIndex, iArrayIndex0, iArrayIndex2, maxIndexUsed,
    maxJ,
    jMaxPower,
    maxDigits, (* arbitrary upper limit on outer loop, to prevent infinite loop *)
    maxTermAddedI, (* max term added for one iteration of the iDigit loop *)
    maxTermAddedJ, (* max term added for one iteration of the jPower loop *)
    maxTermArray,
    iSpecialSum = 0, iDone = 0,
    tiny1, tiny2,
    (* sum = cumulative sum that we are looking for, with exactly n1 d1's, n2 d2's, etc. *)
    requestedSum = 0,
    (* sumOneDigit = requested sum for a single value of iDigit (that is, one power of 10) *)
    sumOneDigit = 0,
    (* sumSmallerK is the cumulative sum over all 0<= k1 <= n1, 0 <= k2 <= n2, etc. *)
    sumSmallerK = 0,
    sumB1, sumB2, term1, term2, bnx2, ajn,
    kList, tableOfLists,
    occurStr,
    timeO, time1, time2
    },
```

```
iPrint = iSumPrintLevel;
(* we use nDecimals because we cannot reset the input parameter nDecimals0 *)
nDecimals = nDecimals0;
If[nDecimals < 5, nDecimals = 5];
nDec = nDecimals + 2;
If[iPrint >= 2,
    Print["iBase = ", iBase, ", digitList = ", digitList, ", countList = ", countList];
    Print["nDecimals = ", nDecimals];
];
timeO = TimeUsed[];
nConditions = Length[digitList];
(* validate all input entered by the user *)
If[ (iBase < 2) || (iBase > 10),
    Print["Base = ", iBase, " must be in the range from 2 through 10."];
    Return[0];
];
If[nConditions < 1,
    Printf[" error 1: nConditions = ", nConditions, " must be at least 1."];
    Return[0];
];
If[nConditions > iBase,
    Print[" error 2: nConditions = ", nConditions, " must be <= base = ", iBase];
    Return[0];
];
If[nConditions != Length[countList],
    Print["Mismatch: digit list and count list have different lengths (", Length[digitList],
            " and ", Length[countList], ")"];
        Return[0];
];
For[i = 1, i <= nConditions, i++,
    If[ (digitList[[i]] < 0) || (digitList[[i]] >= iBase),
        Print["digit # ", i, " = ", digitList[[i]], " is not valid in base ", iBase];
        Return[0];
    ];
    For[j = i+1, j <= nConditions, j++,
        If[digitList[[i]] == digitList[[j]],
            Print["error: digit # ", i, " = ", digitList[[i]], " is duplicated"];
            Return[0];
        ];
```

```
    ]; (* end for j loop *)
    If[countList[[i]] < 0,
        Print["count # ", i, " = ", countList[[i]], " must be O or greater."];
        Return[0];
    ];
]; (* end for i loop *)
(* if every non-zero digit has zero occurrences, then we have an empty sum.
    example: the sum of 1/n where n has no 1 in base 2. just return 0 and
    a message. *)
If[nConditions == iBase-1,
    j = 0;
    For[i = 0, i <= iBase-1, i++,
        If[ (countList[[i]] == 0) && (digitList[[i]] != 0),
            j++;
        ]; (* end If[ ] *)
    ]; (* end for i loop *)
    If[j == iBase-1,
        Print["all non-zero digits have 0 occurrences: this is an empty sum"];
        Return[0];
    ];
]; (* end If[ ] *)
(* at this point, we can assume all input is valid.
    proceed with the calculation. *)
tiny1 = 1/10^(2 * nDec) ;
tiny2 = 1/10^(nDec + 5) ;
(* the larger nDecimals is, the larger maxDigits and jMaxPower need to be.
    in most cases, maxDigits = 60 * nDecimals works.
    maxDigits is used only to make sure the outer loop is finite,
    so we can set it to any reasonable value.
*)
maxDigits = 60 * nDecimals;
If[maxDigits < 500, maxDigits = 500];
(* if nConditions = 1 and countList[[1]] = (say) 434, maxDigits needs to be fairly large *)
If [Max[countList] > 10, maxDigits = maxDigits * 6];
If[iPrint >= 2, Print["maxDigits = ", maxDigits] ];
(* directSumDigits is roughly the log of 1000 to base (iBase).
    so, for base 10, directSumDigits = 3. for base 2, directSumDigits = 10.
*)
directSumDigits = Ceiling[ Log[iBase, 1000] ];
maxJ = computeMaxPowerNeeded[iBase, nDecimals, directSumDigits];
```

```
    If [maxJ < 0,
    maxJ = -maxJ; (* change back to positive, then print it *)
    Print["Could not find good estimate for maxJ; computed maxJ = ", maxJ];
    Return[0];
    ];
    If[iPrint >= 2,
        Print["nDecimals = ", nDecimals, ", directSumDigits = ", directSumDigits,
            ", computed maxJ = ", maxJ];
    ];
jMaxPower = maxJ; (* this value depends on nDecimals *)
(* if all digits 0 through iBase-1 are assigned a maximum number of occurrences,
    then the series is finite. in this case, the number of digits in the
    largest possible denominator is the sum of the countList[[ ]] values.
    example: if we want one each of 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, then the
    (finite) series terminates after 10-digit denominators.
*)
    If[nConditions == iBase,
        iSpecialSum = Sum[countList[[i]], {i, nConditions}]
];
maxTermArray = Table[ 0, { maxJ } ];
(* sjk and sjkPrev are tables (arrays) of size maxJ by maxIndexUsed *)
maxIndexUsed = Product[1 + countList[[i]], {i, nConditions}];
If[iPrint >= 3,
        Print["maxIndexUsed = ", maxIndexUsed];
];
kList = Table[ 0, {nConditions + 1} ];
tableOfLists = Table[ 0, { maxIndexUsed }, { nConditions + 1 } ];
cumulativeSums1 = Table[ 0, { countList[[1]] + 1 } ];
For[i = 1, i <= maxIndexUsed, i++,
        (* for each index value, compute the list and store it in tableOfLists. *)
        kList = getListFromArrayIndex[i, nConditions, countList];
        (* debugging: check that the mapping and its inverse work properly *)
        iArrayIndex = getArrayIndexFromList[nConditions, countList, kList];
        If[iArrayIndex != i,
            Print["error: i = ", i, " != iArrayIndex = ", iArrayIndex];
            Return[0];
```

```
    ];
    For[j = 1, j <= nConditions, j++,
        tableOfLists[[i, j]] = kList[[j]]
    ];
]; (* end for i loop *)
iArrayIndex0 = getArrayIndexFromList[nConditions, countList, countList];
If[iArrayIndex0 != maxIndexUsed,
    Print["error: iArrayIndex0 != maxIndexUsed: iArrayIndex0 = ", iArrayIndex0, ",
            maxIndexUsed = ", maxIndexUsed];
    Return[0];
];
(* initialize sjkPrev and sjk to be 2-dimensional arrays *)
sjk = Table[ 0, { maxJ }, { maxIndexUsed } ];
(* directly compute the sum for denominators having <= (directSumDigits) digits,
    that is, for denominators <= iBase^(directSumDigits) - 1.
*)
For[iDigit = 1, iDigit <= directSumDigits, iDigit++,
    (* set sjkPrev to an array of O's *)
    sjkPrev = Table[ 0, { maxJ }, { maxIndexUsed } ];
    directSummation[iBase, iDigit, jMaxPower, nConditions, digitList, countList, nDec];
    (* requestedSum is the partial sum, so far, of the desired series *)
    requestedSum += sjkPrev[[ 1, iArrayIndex0 ]];
    For[i = 1, i <= maxIndexUsed, i++,
        sumSmallerK += sjkPrev[[ 1, i ]]
    ]; (* end For i loop *)
    If[ (nDigits > 0) && (nDigits == iDigit),
        iDone = 1
    ];
    If[ (threshold > 0) && (requestedSum > threshold),
        Return[ { requestedSum , iDigit } ];
    ];
    If [iDone == 1,
    If[iPrint >= 1,
        If[iBase == 10,
            Print["partial sum through ", iDigit, " digits = ", requestedSum] ,
```

```
        Print["partial sum through ", iDigit, " (base ", iBase, ") digits = ", requestedSum]
    ];
    If[ (nConditions != 1) || (countList[[1]] != 0),
        Print[" partial sum for all ", maxIndexUsed, " 'at most' conditions = ", sumSmallerK]
    ];
    ]; (* If[iPrint] *)
    Break[];
]; (* end if iDone == 1 *)
If[iPrint >= 3,
    Print[iDigit, " digits"]
];
If[iPrint >= 3,
    Print[" partial sum for ", iDigit, " digits = ", sjkPrev[[ 1, iArrayIndex0 ]] ];
    Print[" sum = ", requestedSum]
];
If[iPrint >= 3,
    (* no need for next line if nConditions = 1 and countList[[1]] = 0 *)
    If[ (nConditions != 1) || (countList[[1]] != 0),
        Print[" sum for all ", maxIndexUsed, " 'at most' conditions = ", sumSmallerK]
    ];
];
    If[nConditions == 1, updateCumulativeSums1[iBase, nConditions, countList] ];
]; (* end For iDigit direct summation loop *)
time1 = TimeUsed[];
If[iDone == 1,
    Goto[endLoops]
] ;
If[iPrint >= 2,
    Print["direct sum through ", directSumDigits, " digits = ", requestedSum]
];
iDigitStart = directSumDigits + 1;
(* here is the main loop *)
For[iDigit = iDigitStart, iDigit <= maxDigits, iDigit++, (* outermost loop *)
    (* compute the sum of numbers in the set that have have (iDigit) digits,
            using the sums of reciprocals of powers over numbers having (iDigit-1) digits
    *)
```

```
maxTermAddedI = 0; (* largest term added for this value of iDigit *)
maxTermArray = Table[ 0, { jMaxPower } ]; (* set maxTermArray to all 0's *)
For[jPower = 1, jPower <= jMaxPower, jPower++,
    nMax = jMaxPower - jPower;
    maxTermAddedJ = 0; (* largest term added for this value of jPower *)
    For[iArrayIndex = 1, iArrayIndex <= maxIndexUsed, iArrayIndex++,
        (* for this index value, set kList = corresponding n-tuple. *)
        kList = getListFromArrayIndex[iArrayIndex, nConditions, countList];
        sumB1 = sumB2 = 0;
        (* here is the "infinite" loop, where n goes from O to infinity *)
        For[n = 0, n <= nMax, n++,
            ajn = Binomial[jPower + n - 1, n] / power[iBase, jPower + n];
            If [OddQ[n], ajn = -ajn]; (* this is now (-1)^n * ajn *)
            For[k = 1, k <= nConditions, k++,
                If [kList[[k]] > 0,
                        (* this is for { 10 * S(i, k1, k2, ..., k(k)-1, ..., kn) + d(k) } *)
                        kList[[k]] = kList[[k]] - 1; (* decrement the kth index value *)
                        iArrayIndex2 = getArrayIndexFromList[nConditions, countList, kList];
                term1 = power[digitList[[k]], n] * ajn * sjkPrev[[ jPower + n, iArrayIndex2 ]];
                        sumB1 = sumB1 + term1;
                maxTermAddedJ = Max[maxTermAddedJ, Abs[term1]];
                kList[[k]] = kList[[k]] + 1; (* restore the kth index value *)
            ]; (* end if kList[k] *)
                ]; (* end For k loop *)
            (* this is for the final set in the above union
                { 10 * S(i, k1, k2, ..., kn) + d. where d is none of d1, d2, ..., dn }
                *)
            bnx2 = bn[iBase, n, nConditions, digitList];
            term2 = bnx2 * ajn * sjkPrev[[ jPower + n, iArrayIndex ]];
            maxTermAddedJ = Max[maxTermAddedJ, Abs[term2]];
            sumB2 = sumB2 + term2;
        ]; (* end For n loop *)
        sjk[[jPower, iArrayIndex]] = sumB1 + sumB2;
    ]; (* end For iArrayIndex loop *)
    maxTermArray[[jPower]] = maxTermAddedJ;
    maxTermAddedI = Max[maxTermAddedI, maxTermAddedJ];
]; (* end For jPower loop *)
```

```
sumOneDigit = sjk[[1, iArrayIndex0 ]]; (* sum for this many digits *)
requestedSum += sumOneDigit;
If[ (threshold > 0) && (requestedSum > threshold),
    Return[ { requestedSum , iDigit } ];
];
For[i = 1, i <= maxIndexUsed, i++,
    sumSmallerK += sjk[[1, i]] (* update the sum for all 'at most' conditions *)
];
If[ (nDigits > 0) && (nDigits == iDigit), (* request is to compute partial sums *)
    iDone = 1;
    If[iPrint >= 1,
    If[iBase == 10,
        Print["partial sum through ", iDigit, " digits = ", requestedSum] ,
        Print["partial sum through ", iDigit, " (base ", iBase, ") digits = ", requestedSum]
    ];
    If[ (nConditions != 1) || (countList[[1]] != 0),
        Print[" partial sum for all ", maxIndexUsed, " 'at most' conditions = ", sumSmallerK]
    ];
    ]; (* If[iPrint] *)
    Break[];
];
sjkPrev = sjk; (* this copies the entire array *)
If[nConditions == 1, updateCumulativeSums1[iBase, nConditions, countList] ];
If[iPrint >= 3,
    Print[" partial sum for ", iDigit, " digits = ", N[sumOneDigit, nShow], ",
            total = ", N[requestedSum, nShow] ]
];
If[iPrint >= 4,
    (* no need for next line if nConditions = 1 and countList[[1]] = 0 *)
    If[ (nConditions != 1) || (countList[[1]] != 0),
        Print[" sum for all ", maxIndexUsed, " 'at most' conditions = ", N[sumSmallerK, nShow] ]
    ];
];
If[ (iSpecialSum > 0) && (iDigit > iSpecialSum),
    iDone = 1;
    If[iBase == 10,
        Print["this is a finite series that terminates after ", iSpecialSum, " digits"],
        Print["this is a finite series that terminates after ", iSpecialSum, " digits",
```

```
            " (base ", iBase, ")"]
    ];
];
(* if all terms for a given jPower were very small, then we can decrease jMaxPower *)
If[jMaxPower > 2,
    j1 = jMaxPower;
    For[jPower = jMaxPower, jPower >= 2, jPower--,
        If[maxTermArray[[jPower]] < tiny1,
            j1 = jPower
            Break[]
        ];
    ]; (* end for jPower loop *)
    If[jMaxPower != j1,
        If[iPrint >= 4,
            Print["iDigit = ", iDigit, ": changing jMaxPower from ", jMaxPower, " to ", j1]
        ];
        jMaxPower = j1;
    ];
]; (* end if jMaxPower > 2 *)
If[nDigits > 0,
    Continue[]; (* if calculating partial sums only, skip checks for convergence *)
];
(* if all sjk values are tiny, then we are done *)
iAllTiny = 1;
For[k1 = 0, k1 <= maxIndexUsed, k1++,
    If[sjk[[2, k1 ]] > tiny2,
        iAllTiny = 0; (* not all are tiny *)
        Break[];
    ]
];
If[ (iAllTiny == 1) && (sjk[[1, iArrayIndex0 ]] < tiny2),
    If[(sjk[[1, iArrayIndex0 ]] != 0) && (sjk[[1, iArrayIndex0 ]] / requestedSum < tiny2),
        iDone = 1;
        time2 = TimeUsed[];
        time2 = Round[time2-time0];
        If[iPrint >= 2,
            Print["last iteration of main loop:"];
            Print[" max term added = ", maxTermAddedI, ", sum for ", iDigit,
                        " digits = ", nf[sumOneDigit, nShow]];
        ];
        If[iPrint >= 2,
            Print["iteration done after ", iDigit, "-digit denominators (", time2, " seconds)."]
        ];
    ]; (* end If sjk ... *)
```

```
    ]; (* end If iAllTiny ... *)
    If [ (iDigit == maxDigits) && (iDone == 0),
        Print["last iteration (", maxDigits, ") of main loop, but no convergence yet."];
        Print[" suggestion: make 'maxDigits' (now = ", maxDigits, ") larger,"];
        Print[" read the file in again, and start over."];
        Print[" partial sum for ", iDigit, " digits = ", N[sjk[[1, iArrayIndex0 ]], nShow] ];
        requestedSum = 0;
        Break[]
    ];
    If[ (iDone == 1) && (iPrint > 0),
        Print["sum = ", nf [requestedSum, nDecimals, iFormatted] ];
        If[ (nConditions == 1) && (countList[[1]] != 0),
            printSumsWithKOccurrences[nDecimals, nConditions, digitList, countList, iFormatted];
            If[countList[[1]] == 1, occurStr = "occurrence ", occurStr = "occurrences"];
            Print[" sum with at most ", maxIndexUsed - 1, " ",
                        occurStr, " of ", digitList[[1]], " = ",
                        nf[sumSmallerK, nDecimals, iFormatted] ];
            ];
            If[nConditions > 1,
                printAtMostSumNCond[nConditions, digitList, countList,
                                    sumSmallerK, nDecimals, iFormatted];
            ];
        Break []
];
    ]; (* end For iDigit loop *)
Label[endLoops];
    (* return just the value of the requested sum *)
    nf[requestedSum, nDecimals, iFormatted]
]; (* end of private function computeIrwinSum *)
(* here are the functions that the user can call:
    iSum, iSumFormatted, iPartialSum, iPartialSumThreshold, setPrintLevel, getPrintLevel *)
iSum[digitList_?VectorQ, countList_?VectorQ, nDecimals_:15, iBase_:10, iFormatted_:0] :=
Module[
    (* examples:
            iSum[{ 9 }, { 3 } ] = sum of 1/n where n has exactly three 9's.
            iSum[ { 9 }, { 3 }, 30 ] = same calculation, display result rounded to 30 decimals.
```

```
        iSum[{9, 3}, {2, 0}] = sum of 1/n where n has two 9's and no 3's
                        =2.593253652747189 .
    *)
    { },
    computeIrwinSum[iBase, digitList, countList, nDecimals, iFormatted]
]; (* end of iSum[digit list, count list, decimals, base] *)
iSum[d_Integer, iCount_Integer, nDecimals_:15, iBase_:10, iFormatted_:0] :=
Module[
    (* examples:
            iSum[ 9, 0 ] = sum of 1/n where n has no 9's.
            iSum[ 9, 2 ] = sum of 1/n where n has two 9's.
            iSum[ 9, 2, 30 ] = same calculation, to 30 decimals.
    *)
    { },
    (* just call the "list" version of iSum *)
    iSum[ { d }, { iCount }, nDecimals, iBase, iFormatted ]
]; (* end of iSum[digit, count, decimals, base] *)
iSumFormatted[digitList_?VectorQ, countList_?VectorQ, nDecimals_:15, iBase_:10] :=
Module[
    { iFormatted = 1 },
    iSum[digitList, countList, nDecimals, iBase, iFormatted]
]; (* end of iSumFormatted[digit list, count list, decimals, base] *)
iSumFormatted[d_Integer, iCount_Integer, nDecimals_:15, iBase_:10] :=
Module[
    { iFormatted = 1 },
    iSum[d, iCount, nDecimals, iBase, iFormatted]
]; (* end of iSumFormatted[digit, count, decimals, base] *)
setPrintLevel[i_Integer:1] :=
Module[
    { i2 },
    i2 = i;
    (* O is the minimum value a user can set. however, some
        functions like iPartialSumThreshold can set it lower *)
    If[i2 < 0, i2 = 0, Null];
    iSumPrintLevel = i2;
    Print["print level set to ", iSumPrintLevel]
]; (* end of setPrintLevel *)
```

```
getPrintLevel[i_Integer:1] :=
Module[
    { },
    Print["current print level is ", iSumPrintLevel]
]; (* end of getPrintLevel *)
iPartialSum[digitList_?VectorQ, countList_?VectorQ, nDigits_Integer?Positive,
                    nDecimals_:15, iBase_:10] :=
Module[
    (* examples of "list" version of iPartialSum:
                iPartialSum[ {9}, {0}, 30 ] = sum to 10^30 of 1/n where n has no 9's.
                iPartialSum[ { 9, 0 }, { 3, 1 }, 20 ] = sum to 10^20 of 1/n where n has three 9's and one 0.
    *)
    { iFormatted = 0 },
    computeIrwinSum[iBase, digitList, countList, nDecimals, iFormatted, nDigits]
]; (* end of iPartialSum[digit list, count list, numDigits, decimals, base] *)
iPartialSum[d_Integer, iCount_Integer, nDigits_Integer?Positive, nDecimals_:15, iBase_:10] :=
Module[
    (* examples:
            iPartialSum[ 9, 0, 20 ] = sum to 10^20 of 1/n where n has no 9's.
            iPartialSum[ 9, 2, 20 ] = sum to 10^20 of 1/n where n has two 9's.
            iPartialSum[ 9, 2, 20, 30 ] = same calculation, to 30 decimals.
            iPartialSum[1, 1, 6, 15, 2] = partial sum of 1/n where n has one 1
                    in base 2, for n < 2^6, to 15 decimals.
    *)
    { },
    (* just call the "list" version of iPartialSum *)
    iPartialSum[ { d }, { iCount }, nDigits, nDecimals, iBase ]
]; (* end of iPartialSum[digit, count, numDigits, decimals, base] *)
iPartialSumThreshold[digitList_?VectorQ, countList_?VectorQ, threshold_?Positive,
                nDecimals_:15, iBase_:10] :=
Module[
    (* example of "list" version of iPartialSum: the sum of 1/n where n has three 9's and one 0, is
            iSum[{9, 0}, {3, 1}] = 2.888545932755274 . therefore, the threshold must be less than this
            value. in this example, we take the threshold to be 2.
            iPartialSumThreshold[ { 9, 0 }, { 3, 1 }, 2 ]
                = {27, 1.910422503190251, 28, 2.0043388417551473}
    *)
    { iFormatted = 0, nDigits = 0, totalSum, iPrintLevelSave,
        xSum1 = 0, nDig1 = 0, xSum2, nDig2, errorReturn = {-1, -1, -1, -1}, tAcc, nDec2,
```

```
    errStr = "Use backquote notation iPartialSumThreshold[ digit, count, threshold``nDecimals] ],\
or enclose the threshold in double quotes"
    },
tAcc = 1 + Floor[Accuracy[threshold]];
If[tAcc == Infinity,
        nDec2 = nDecimals,
        nDec2 = Max[nDecimals, tAcc]
];
iPrintLevelSave = iSumPrintLevel;
iSumPrintLevel = -1; (* suppress all printed output except error messages *)
totalSum = iSum[digitList, countList, nDec2, iBase];
If[threshold > totalSum,
        Print["Error: your threshold is greater than the sum of the entire series."];
        iSumPrintLevel = iPrintLevelSave;
        Return[ errorReturn ] (* error *)
];
If[threshold == totalSum,
        Print["Error: your threshold is very close to the sum of the entire series.",
            " You need more accuracy. ", errStr];
        iSumPrintLevel = iPrintLevelSave;
        Return[ errorReturn ] (* error *)
];
{ xSum2 , nDig2 } = computeIrwinSum[iBase, digitList, countList, nDec2,
                                    iFormatted, nDigits, threshold];
If[nDig2 > 1,
        nDig1 = nDig2 - 1;
        xSum1 = iPartialSum[digitList, countList, nDig1, nDec2, iBase];
];
If[ ( ! (xSum1 < threshold)) && (nDig2 > 1),
        (* xSum1 should be less than the threshold.
            if it is not, decrease the number of digits by 1 *)
        nDig2 = nDig1;
    xSum2 = xSum1;
    nDig1 = nDig2 - 1;
    xSum1 = iPartialSum[digitList, countList, nDig1, nDec2, iBase];
];
iSumPrintLevel = iPrintLevelSave;
(* if the computed xSum1 and xSum2 are correct, then
    xSum1 must be < threshold and xSum2 must be >= threshold. *)
If[ ! (xSum1 < threshold),
    Print["Not enough accuracy [1]. ", errStr];
    Return[ errorReturn ] (* error *)
];
```

```
    If[ ! (xSum2 >= threshold),
    Print["Not enough accuracy [2]. ", errStr];
    Return[ errorReturn ] (* error *)
    ];
    { nDig1, xSum1, nDig2, xSum2 }
]; (* end of iPartialSumThreshold[digit list, count list, threshold, decimals, base] *)
iPartialSumThreshold[d_Integer, iCount_Integer, threshold_?Positive, nDecimals_:15, iBase_:10] :=
Module[
    (* examples:
        1. iPartialSumThreshold[9, 1, 23]: for the sum of 1/n where n has one 9,
                whose sum is iSum[9, 1] = 23.044287080747848, iPartialSumThreshold[9, 1, 23]
                computes about how far we need to go to reach a partial sum of 23.
            the output is {80, 22.995762680948152, 81, 23.000125707332644}
            interpretation of output:
                the partial sum of 1/n through n < 10^80 is 22.99576..., which is less than 23.
                however, if we include all terms through 81 digits (that is, n < 10^81),
                then the partial sum is 23.000125707332644, greater than your threshold.
            now that we have the 80 and 81, we can use iPartialSum to verify this result:
                iPartialSum[9, 1, 80] = 22.995762680948152
                iPartialSum[9, 1, 81] = 23.000125707332644 .
        2. iPartialSumThreshold[9, 1, 23, 30]: same calculation, to 30 decimals.
        3. iSum[1, 1, 15, 2] = 2 = sum (to 15 decimals) of 1/n where n has one 1 in base 2.
        iPartialSumThreshold[1, 1, 1.99, 15, 2] = point at which above series reaches 1.99:
                {7, 1.984375000000000, 8, 1.9921875000000000}
            iPartialSum confirms this result:
                iPartialSum[1, 1, 7, 15, 2] = 1.984375000000000
                iPartialSum[1, 1, 8, 15, 2] = 1.992187500000000 .
            4. iPartialSumThreshold[9, 1, 23.044287080747] fails because the threshold is too close
            to the sum of the entire series. the best solution is to use Mathematica's
            backquote notation to increase the accuracy of the threshold. 23.044287080747``25 =
            23.044287080747000000000000. the following gives the correct answer:
                iPartialSumThreshold[9, 1, 23.044287080747``25]
            returns
                {327, 23.04428708074693636344610077, 328, 23.04428708074702511802366170} .
            you can also use double quotes to convert the threshold to a string:
                iPartialSumThreshold[9, 1, "23.044287080747"]
            also returns the correct result
                {327, 23.044287080746936363, 328, 23.044287080747025118} .
    *)
    { },
    (* call the list version of this function *)
    iPartialSumThreshold[ { d }, { iCount }, threshold, nDecimals, iBase ]
```

```
]; (* end of iPartialSumThreshold[digit, count, threshold, decimals, base] *)
```

```
iPartialSumThreshold[digitList_?VectorQ, countList_?VectorQ, pSumStr_String,
    nDecimals_:15, iBase_:10] :=
Module[
    (* the threshold was entered as a string. add double backquotes to specify the accuracy,
        then call another version of iPartialSumThreshold. *)
    { pSum, inputStr2, decPtList, quoteList, nDecimalsInput, nDec2,
        errorReturn = { -1, -1, -1, -1 }
    },
    decPtList = StringPosition[pSumStr, "."]; (* get list of starting, ending positions *)
    quoteList = StringPosition[pSumStr, "`"];
    If[ (Length[decPtList] == 1) && (Length[quoteList] == 0),
        (* input has one decimal point and no backquotes (usual case); number of
            decimals = number of chars between decimal point and end of string.
            append backquotes and a number to specify the accuracy of the input,
            then convert the new string to a floating-point number with that accuracy.
        *)
        nDecimalsInput = StringLength[pSumStr] - decPtList[[1]][[1]] ;
        (* nDec2 = Max[nDecimalsInput + 2, nDecimals]; *) (* from kempnerSums.m *)
        nDec2 = Max[nDecimalsInput + 5, nDecimals];
        inputStr2 = pSumStr <> "`-" <> ToString[nDec2] ;
        pSum = ToExpression[inputStr2];
        ,
        (* otherwise (backquotes already in use, or input is an integer):
            just convert the input string to a number *)
        pSum = ToExpression[pSumStr]
    ];
    If[ (pSum == $Failed) || (NumericQ[pSum] == False),
        Print["Invalid input"]; (* cannot convert input to a numeric value *)
        Return[ errorReturn ]
    ];
    iPartialSumThreshold[digitList, countList, pSum, nDecimals, iBase]
```

]; (* end of iPartialSumThreshold[digit list, count list, threshold string, decimals, base] *)
iPartialSumThreshold[d_Integer, iCount_Integer, pSumStr_String, nDecimals_:15, iBase_:10] :=
Module[
\{ \},
iPartialSumThreshold[ \{ d \}, \{ iCount \}, pSumStr, nDecimals, iBase]
]; (* end of iPartialSumThreshold[digit, count, threshold string, decimals, base] *)

End [ ] (* end the private context *)
(* protect the exported symbols *)
Protect[iSum, iSumFormatted, iPartialSum, iPartialSumThreshold, setPrintLevel, getPrintLevel];

EndPackage [ ] (* end the package context *)

## 8. The Mathematica Code for Kempner Sums

```
(* :Title: kempnerSums.m -- a package to compute sums of Kempner series *)
(* :Mathematica Version: this works in versions 4.0, 5.2, 6.0, 7.0, 8.0, and 9.0 *)
(* :Package Version: 2.3.
    8/25/2015: reformatted some lines to make them shorter for LaTeX.
    12/31/2012: new version of nf[ ] function.
    Version 2.2:
    7/24/2008: updated links to paper and on-line version of this file.
    5/30/2008: removed old code that is no longer used.
    5/14/2008: two minor changes to make this be compatible with
    mathematica versions 4 and 5.2.
    5/29/2008: use "===", not "==", to check expression types.
    reworked some of the examples.
*)
(* :Authors: Robert Baillie and Thomas Schmelzer *)
(* :Copyright: none. This code is in the public domain.
    Users are encouraged to improve this code.
*)
(* :Summary:
    This package computes sums of "Kempner" series.
    The harmonic series 1/1 +1/2 +1/3 + .. + 1/n + .. diverges. This means
    that the sum can be made as large as desired by adding enough terms.
    Suppose we delete from this series all terms with a "9" in the denominator.
    That is, we remove 1/9, 1/19, 1/29, etc. Then the remaining series converges
    to a sum less than 80. This was first proved in 1914 by A. J. Kempner.
    We can also delete from the harmonic series all terms whose denominators
    contain:
        any digit 0, 1, ..., 9;
        or
        a list of one or more digits, such as the even digits {0, 2, 4, 6, 8};
        or
        any individual multi-digit string or number, such as "00", 42, or 314159;
        or
        any list of patterns of digits, such as { 3, "00", 42, 314159 }.
    In all such cases, the terms that remain form a convergent series.
    However, these series converge much too slowly to get even a rough
    estimate of the sum simply by adding up the terms. For example,
    for the series with 9 removed, the sum of the first 10^27 terms
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    still differs from the final sum (22.92067...) by about 1.
    This package computes the sums of these series to high precision.
*)
(* :Keywords: kempner, harmonic series *)
(* :Sources:
    The original article that proves convergence of these series is:
        A. J. Kempner, "A Curious Convergent Series",
        American Mathematical Monthly, vol. 21, pages 48-50 (1914).
    The original algorithm that shows how to compute a few of these sums is in:
    Robert Baillie, "Sums of Reciprocals of Integers Missing a Given Digit",
        American Mathematical Monthly, vol. 86, pages 372-374 (May, 1979).
    The functions in this file implement the algorithm in the following article:
        Thomas Schmelzer and Robert Baillie, "Summing a Curious, Slowly Convergent
        Series", in the American Mathematical Monthly, vol. 115, pages 525-540
        (June/July 2008).
        The latest version of this Mathematica code may be downloaded from
        http://library.wolfram.com/infocenter/MathSource/7166/
*)
BeginPackage[ "kempnerSums`" ]
(* :Examples:
    The main function in this package is kSum.
    kSum[X] computes the sum of 1/n where n has no digits in the set X.
    The output is rounded to the default number of decimal places. This
    default is initially set to 15, but you can use kSumSetDefaultDecimals[m]
    to reset the default to m decimal places.
    Examples:
        kSum[9] computes the sum of 1/n where n has no digit 9.
        This sum is about 22.920676619264150 .
    The second parameter is the number of decimal places:
        kSum[9, 20] = 22.92067661926415034816 .
    kSum can also take a third parameter (the base).
        kSum[X, m, b] computes the sum of 1/n where n has no digits in X in base b.
        The sum is rounded and displayed to m decimal places.
        If the third parameter is omitted, the base is assumed to be 10.
        If you want to use the third parameter to specify a base other than 10,
        then you must include all three parameters:
            kSum[0, 20, 2] = 1.60669515241529176378 .
        The first parameter, X, is the set of numbers that are to be omitted
```


## from the denominators.

X can be:
a single number: kSum [0] or kSum [314]
or
a list of numbers and/or strings enclosed in braces: kSum [ \{ 1, "07", 32 \} ].

```
kSum[{3, 1, 4}] computes the sum of 1/n where n contains no 3, no 1,
and no 4 in base 10. The sum is about 3.769948379731421 .
kSum[314] computes the sum of 1/n where n contains no 314 in base 10.
The sum is about 2299.829782767518338.
If a missing digit pattern is an integer with more than one digit, and
it begins with a 0, then you must enclose the number in quotes. This
is because Mathematica will interpret the input 09 as the integer 9,
which gives a different result:
    kSum[09] = 22.920676619264150,
    but
    kSum["09"] = 230.404757005814285 .
kSum[ { 0, 2, 4, 6, 8, 55, 13579 } ] computes the sum of 1/n where
n has no even digit, no 55, and no 13579. This example is in the
Schmelzer-Baillie paper. The sum is about 3.090849149653806 .
kSum[0, 30, 2] computes the sum of 1/n where n has no 0 in base 2.
In order to specify the base as the third parameter, we cannot omit
the second parameter (the number of decimal places).
Note: this sum is equal to the sum of the rapidly-converging series
1/1 + 1/3 + 1/7 + 1/15 + 1/31 + . . =
Sum[1/(2^k-1), {k, 1, Infinity}] = 1.606695152415291763783301523191 ...,
This example provides an independent check on the algorithm here.
```

kSumFormatted is just like kSum, except that kSumFormatted formats
the output digits into groups of 5:
kSumFormatted[9] = 22.920676619264150 .
(Technical notes: the output from kSumFormatted is not a number, but is
instead an object of type "NumberForm". Also, remember that Mathematica
will interpret the input 22.9206766192 as 22.92067 * 66192).

Once you have computed a sum, you can use kPartialSum to compute partial sums of the same series. kPartialSum uses the most recent sum computed by kSum or kSumFormatted.
For example, immediately after computing
$\mathrm{kSum}[9]=22.920676619264150$,
you can compute the partial sum of the same series over all terms whose denominators have at most 30 digits (that is, all denominators in the series that are less than 10~30):
kPartialSum[30] gives 21.971055078178619 .
This shows that even if it were possible to use brute force to add all terms of the series up through 30 digits (more than $10^{\wedge} 27$ terms), that partial sum would still differ from the sum of the entire series by almost 1.0 .

Here is an example in base 2:
$\mathrm{kSum}[0,20,2]=1.60669515241529176378$.
Here is the partial sum over all denominators in the series that have at most 5 digits in base 2, that is, denominators at most (decimal) 31 . kPartialSum [5] = 1.575115207373272
This partial sum is just $1+1 / 3+1 / 7+1 / 15+1 / 31=1709 / 1085$.

You can specify a different number of decimal places in the second parameter.
After computing kSum [9]:
$\mathrm{kPartialSum}[30,20]=21.97105507817861948711$.
The next two calculations show that in order to make the partial sum exceed 22.9, you would need to include at least some terms with denominators of 67 digits:
kPartialSum [66] $=22.899283165770095$
kPartialSum [67] $=22.901422511119500$.

If we exclude denominators containing 314159, the Kempner sum is 2302582.33386... .
At some point before this, the partial sum will first exceed 1000000.
How far do we have to go in the series before this happens?
kPartialSumThreshold gives us the answer. kPartialSumThreshold gives us the (approximate) point at which a partial sum exceeds a given threshold, and shows how extraordinarily slowly these series converge to their final sums.

After computing kSum[314159] = 2302582.333863782607892, we call
kPartialSumThreshold[1000000]. The result is:
\{569679, 569680, 999999.793917693872470, 1.000001096506746783363*10~6\}
These four numbers tell us that the sum of the series through all
denominators having 569679 digits is 999999.793917693872470 , just under
1000000, while the sum through denominators having 569680 digits is
$1.000001096506746783363 * 10^{\wedge} 6=1000001.096506746783363$, just over 1000000.
In other words, to make the partial sum exceed 1000000 , we would have to include all terms with denominators having 569679 digits, and some terms with denominators having 569680 digits.

These results can be checked with kPartialSum:
$\mathrm{kPartialSum}[569679]=999999.793917693872470$
$\mathrm{kPartialSum}[569680]=1000001.096506746783363$

The next calculation shows that, to make the partial sum exceed 2000000 (that is, to approximate the final sum to just one significant digit), we need to include at least some terms whose denominators have 2029422 digits:
kPartialSumThreshold[2000000] returns
2029421, 2029422, 1.999999789743530856418*10^6, $\left.2.000000092327587901494 * 10^{\wedge} 6\right\}$

The next calculation shows that, to make the partial sum reach the integer part
of the sum of the entire series (that is, to reach 2302582), we must include at least some terms that have denominators of 15746478 digits: kPartialSumThreshold[2302582] returns $\left\{15746477,15746478,2.302581999999948711319 * 10^{\wedge} 6,2.302582000000282576822 * 10^{\wedge} 6\right\}$

Here is an example with an even larger sum. If we exclude denominators
that have the digit pattern "1234567890", the sum is $\mathrm{kSum}[1234567890,20]=2.302585092202703549829506732687 * 10^{\wedge} 10$.
After including some terms with 434304 digits, the sum finally reaches 1 million: kPartialSumThreshold[1000000] returns \{434303, 434304, 999998.47617794443909902633, 1.00000077866303886642136774*10~6\}
After including some terms with 444007324 digits, the sum reaches 1 billion: \{444007323, 444007324, $9.9999999815290203462491699527 * 10^{\wedge} 8$, $\left.1.00000000035548712899465692756 * 10^{\wedge} 9\right\}$

The partial sum over all terms having at most one billion digits is $\mathrm{kPartialSum}\left[10^{\wedge} 9\right]=2.191199428160751825211987 * 10^{\wedge} 9$,
which is still only about $1 / 10$ the sum of the entire series.

You can display the current default number of decimal places with kSumShowDefaultDecimals (no parameters). You can change the default number of decimal places. For example, kSumSetDefaultDecimals[20]
will set the default number of decimal places to 20. In any individual calculation, you can also specify how many decimal places you want.

```
This loop computes the 10 sums that arise when terms with the digits
0, 1, 2, ..., 9 are removed from the harmonic series:
    Do[Print[ i, " ", kSum[i]], {i, 0, 9}]
This computes sums involving all pairs of missing digits, base 10:
    For[i = 0, i <= 9, i++,
        For[j = i+1, j <= 9, j++,
            Print[ i, ",", j, ": ", kSum[{ i, j }] ]
        ]
    ]
```

The next loop prints out the sums for missing digit patterns 00 through 99. For "00", for example, we would delete $1 / 100,1 / 200, \ldots$ from the harmonic series. We need IntegerString[ ] for the input strings "00" - "09" because they have leading zeros. (IntegerString also works with the the remaining digit patterns 10 - 99). These sums range between approximately 220 and 253.

Do[Print[IntegerString[i, 10, 2], " ", kSum[IntegerString[i, 10, 2] , 10]], \{i, 0, 99\}]

The authors believe that all output digits for $k S u m[X, m]$ are correct, rounded in the last place. To verify that you got as many correct decimal places as you wanted, you can run the calculation again with, say, 5 more decimal places.
*)


```
    kSum[314] = 2299.829782767518338\r\n
    kSum[314159, 20] = 2302582.33386378260789202376\r\n
    kSum[{0, 2, 4, 6, 8}] = 3.171765473415905 (denominators with no even digits)\r\n
    kSum[{1, 3, 5, 7, 9}] = 1.962608412994617 (denominators with no odd digits)\r\n
    kSum[{0, 1, 4, 6, 8, 9}] = 1.857333779940978 (denominators with only prime digits)\r\n
    kSum[{\"00\", 11, 22, 33, 44, 55, 66, 77, 88, 99}] = 23.382957488301063 (no double digits)\r\n
    kSum takes an optional third parameter: the base. If not specified, base 10 is used.
    If you use the third parameter, you cannot omit the second parameter.
    Example: kSum[0, 15, 2] = 1.606695152415292 (denominators with no O in base 2).\r\n
    Another form: kSum[ T ], where the input is the T matrix; see the Schmelzer-Baillie paper."
kSumFormatted::usage =
    "Similar to kSum, but the output decimal places are separated into groups of 5 digits."
kPartialSum::usage = "kPartialSum[ nDigits, nDecimals ]\r\n
    Using the most recent input to kSum or kSumFormatted, compute the sum of all terms\
    whose denominators have at most (nDigits) digits.
    You must call kSum or kSumFormatted before calling kPartialSum.\r\n
    Examples:
        kSum[9] = 22.920676619264150 .
        Then, kPartialSum[30] gives 21.971055078178619 .
        This is the sum of all terms in the series with denominators < 10^30.
        Likewise, kPartialSum[31] gives 22.066017232287173 . Therefore, to make the partial\
    sum exceed 22, we need some terms whose denominators have 31 digits.
    This fact can also be established with the function 'kPartialSumThreshold':
        kPartialSumThreshold[22] returns {30, 31, 21.971055078178619, 22.066017232287173}.
    The next two calculations show that, to make the sum exceed 22.9, we need at least some terms\
    whose denominators have 67 digits.
        kPartialSum[66] = 22.899283165770095
        kPartialSum[67] = 22.901422511119500 ."
```

kPartialSumThreshold::usage = "kPartialSumThreshold[s]\r\n
Using the most recent input to kSum or kSumFormatted, compute the number of digits needed
in the denominators to make the partial sum exceed $s$.
The value of $s$ must be less than the sum of the series. If $s$ is a floating-point value close to the
sum of the entire series, you should enclose it in quotes, or use Mathematica's backquote notation to
specify the accuracy of your input. For example, instead of kPartialSumThreshold[22.920676619264149],
you should enter either kPartialSumThreshold[\"22.920676619264149\"] or
kPartialSumThreshold[22.920676619264149`'15]. If you don't do this, you may get a wrong answer.
Four numbers are returned: \{ d1, d2, s1, s2 \}. d2 is the number of digits required in the
denominators to make the partial sum >s. $\mathrm{d} 1=\mathrm{d} 2-1 . \mathrm{s} 1$ and s 2 are the partial sums for d 1 and d 2 .
Given s, s1 and s2 will normally be such that s1 < s < s2.
You must call kSum or kSumFormatted before calling kPartialSumThreshold. \r \n
Example 1:
kSum [9] $=22.920676619264150$.
Then, kPartialSumThreshold[22] returns \{30, 31, 21.971055078178619, 22.066017232287173\}.
This means that, to make the partial sum exceed 22 , we need at least some terms whose denominators $\backslash$
have 31 digits. $\backslash r \backslash n$

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Example 2:
    This example shows what can go wrong when your input is a floating-point number that is
    near the sum of the entire series.
    kSum[9, 20] = 22.92067661926415034816.
    kPartialSumThreshold[22.920676619264149] returns
        {359, 360, 22.92067661926414950999, 22.92067661926414959380} .
    This result is wrong: note that both partial sums are greater than your input value.
    (The result is wrong because Mathematica may slightly altered your floating-point input.)
    To get the correct answer when your input value is close to the sum of the entire series,
    you must specify your input more accurately.
    You can do this using Mathematica's backquote notation:
    kPartialSumThreshold[22.920676619264149``15] returns the correct input:
        {354, 355, 22.92067661926414892870, 22.92067661926414907065}.
    You will also get the correct answer if you enter your input as a string:
        kPartialSumThreshold[\"22.920676619264149\"]."
kSumGetT::usage = "kSumGetT[X] computes and displays the T matrix associated with X.\r\n
    X is the collection of numbers to omit from the denominators.\
    This function takes an optional second parameter: the base. If not specified, base 10 is used."
kSumGetA::usage = "kSumGetA[X] computes and displays the A matrix associated with X.\r\n
    X is the collection of numbers to omit from the denominators.\
    This function takes an optional second parameter: the base. If not specified, base 10 is used."
kSumShowA::usage = "kSumShowA[ ] shows the A matrix for the previously-entered input string or list."
kSumTimeAndMemory::usage = "kSumTimeAndMemory[nDigits] estimates time and memory requirements\
    for kSum[9, nDigits].\
    Note that kSum[99, nDigits] needs more time and memory than kSum[9, nDigits]."
kSumSetDefaultDecimals::usage = "kSumSetDefaultDecimals[n] sets the default number of decimal places\
    to n. Initially, the default is 15."
kSumShowDefaultDecimals::usage =
    "kSumShowDefaultDecimals displays the current default number of decimal places."
Begin["`Private`"] (* begin the private context (implementation) *)
Off[ General::spell1 ];
Off[ General::spell ];
nDefaultDecimals = 15;
getDigitsforNFunction[y_, nDec_] :=
Module[
(* we want to display an extended-precision number to (nDec) decimal places,
    not counting the number of leading zeros if the number is between 0 and 1.
```

```
    if Abs[y] < 1, N[y, nDec] displays the correct number of decimal places;
    otherwise, in N[y, nDec], the number of digits shown includes digits to the left
    of the decimal point; in this case, we must increase the number of digits sent to N[ ].
*)
    { nExtra },
    If [Abs[y] >= 1,
        nExtra = 1 + Floor[ Log[10, Abs[y]] ],
        nExtra = 0
    ];
    nExtra + nDec
] (* end of getDigitsforNFunction *)
(* nf formats a number to (nDec) decimal places, with a space every 5 digits *)
(*
nf[y_, nDec_] :=
    If[y == 0, 0,
        NumberForm[y, Max[ 1, Floor[1+Log[10, Abs[y]]] ] + nDec, DigitBlock->5, NumberSeparator->{""," "}]
    ];
*)
nf[y_, nDec_] :=
    If[y == 0, 0,
        NumberForm[y, getDigitsforNFunction[y, nDec], DigitBlock->5, NumberSeparator->{""," "}]
    ];
```

```
removeDups[inputList_?VectorQ] :=
```

removeDups[inputList_?VectorQ] :=
Module[
Module[
(* remove duplicates from a list without changing the order;
(* remove duplicates from a list without changing the order;
(Union[ ] removes duplicates, but also sorts the list) *)
(Union[ ] removes duplicates, but also sorts the list) *)
{ },
{ },
elimReps[u_, v_] := If[MemberQ[u, v], u, Append[u, v]];
elimReps[u_, v_] := If[MemberQ[u, v], u, Append[u, v]];
Fold[elimReps, {First[inputList]}, Rest[inputList]]
Fold[elimReps, {First[inputList]}, Rest[inputList]]
];
];
(* Mathematica version 4 does not have a Norm[x, y] function. so, to make this package
work in version 4, we define a function that is equivalent to Norm[v, 1], which is:
Norm[v, 1] = Abs[ v[[1]] ] + Abs[ v[[2]] ] + ... + Abs[ v[[n]] ] *)
normOne[v_?VectorQ] := Apply[Plus, Map[Abs, v] ];
(* power2 is just like Power[], but power2 returns 0^0 = 1 with no warning messages *)
power2[0, 0] := 1; power2[x_, y_] := Power[x, y] ;
(* speedMode = 0 for (slow, use less memory), 1 for (fast, lots of memory); 1 is the default *)
(* speedMode = 1; *)

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(* these global variables are computed in kSumX, and are used in kPartialSum *)
A = Null;
extraPol = 0;
n = 0;
Psi = Null;
iBaseSave = 0;
kSumSave = 0;
nDecimalsSave = 0;
Unprotect[kSum, kSumFormatted, kPartialSum, kSumGetT, kSumGetA, kSumShowA];
Unprotect[kSumTimeAndMemory, kSumSetDefaultDecimals, kSumShowDefaultDecimals];
Unprotect [kPartialSumThreshold];
Clear[kSum, kSumFormatted, kPartialSum, kSumGetT, kSumGetA, kSumShowA];
Clear[kSumTimeAndMemory, kSumSetDefaultDecimals, kSumShowDefaultDecimals];
Clear[setT, kSumX];
Clear[convertInputListToStrings];
convertInputListToStrings[inputList_?VectorQ] :=
Module[
(* convert input list to all strings; if a list element is neither a string
nor an integer, return an empty list. *)
{ len, i, str, outputList = { }, bSet },
len = Length[inputList];
For[i = 1, i <= len, i++,
bSet = False;
If[StringQ[ inputList[[i]] ], str = inputList[[i]]; bSet = True];
If[IntegerQ[ inputList[[i]] ], str = ToString[ inputList[[i]] ]; bSet = True];
If[bSet == False,
Print["Error: Invalid input list: value \#", i ," is neither a String nor an Integer."];
Return[ {} ]
];
AppendTo[outputList, str];
];
outputList
];
setT[stringList_?VectorQ, iBase_:10] :=
Module[ (* enter with a list of strings {s1, s2, ...}, like { "12", " "345" }; return the T matrix *)
(* note: this function returns a 2-dimensional matrix, not a list. even when the output
matrix has only one row, it is a matrix of the form {{1,2,3. ...}}, not a vector of the
form {1,2,3, ...}. *)
(* programming note: this function was not written in mathematica's efficient "functional" style.
this function does not need to be very efficient. this function sets up the T matrix farily,
quickly. calculating the sum takes far longer. readers are welcomed to improve this code.
*)
{ nInputStrings, i, s, s2, len, validDigits, j, t, iRow, firstNChars, d, dTest, found, jStart,

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    inputList0, inputList = {}, (* copy of input list *)
    nSingleDigits = 0, nRows = 0, firstRow, nSetStrings = 0, setStrings = {},
    firstJChars, bFound, k, dString, s2Appended, maxMatched, nCompare, iSaved, t2 },
    (* preliminaries: check input validity, make a list of input strings with spaces removed. *)
If[( iBase < 2) || (IntegerQ[iBase] == False),
Print["Error: Base = ", iBase," is not valid."];
t = Table[-1, {1}, {1}]; (* return this invalid 1 by 1 matrix *)
Return[t]
];
inputListO = removeDups[stringList]; (* remove duplicate strings, but keep the order unchanged *)
nInputStrings = Length[inputList0];
If [nInputStrings < 1,
Print["Error: No input string specified."];
t = Table[-1, {1}, {iBase}]; (* return this invalid 1 by iBase matrix *)
Return[t]
];
If[iBase <= 10, (* make a list of all valid digits in this base *)
validDigits = CharacterRange["0", ToString[iBase-1]]
];
(* if iBase > 10, how should we check that the digits are valid?
perhaps we could use lower-case letters a-z for bases 11-36,
as Mathematica does for BaseForm[ ]. note: we can still enter
a digit in any base, say 0 in base 100 or base 1000. for example,
kSum[0, 10, 100] = 460.5252026385
and
kSum[0, 10, 1000] = 6907.756101048 . *)
For[i = 1, i <= nInputStrings, i++,
(* make sure this list element is an integer or a string *)
If[UnsameQ[ Head[inputList0[[i]]] , String ] \&\& UnsameQ[ Head[inputList0[[i]]] , Integer ],
Print["Error: Invalid input list: value[",i,"] is not a String."];
If [Head[inputList0[[i]]] === Symbol, Print["Two commas in a row."] ];
t = Table[-1, {1}, {iBase}]; (* return this invalid (1 by iBase) matrix *)
Return[t]
];
s = inputList0[[i]];
If [Head[s] === Integer, s = ToString[s]];
(* for convenience, user can include spaces, but we ignore them *)
s2 = StringReplace[s, " " -> ""];
len = StringLength[s2];
If[len == 0, Continue[]];

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    If[iBase <= 10, (* verify that this input string is a valid number in this base *)
        For[j = 1, j <= len, j++,
            If[MemberQ[validDigits, StringTake[s2, {j}]] == False, (* jth character *)
                Print["Error: input character '", StringTake[s2, {j}],
                    "' is not a valid digit in base ", iBase];
                t = Table[-1, {1}, {iBase}]; (* return this invalid (1 by iBase) matrix *)
                Return [t]
            ];
        ];
    ];
    (* this string is ok, so add it to inputList *)
    AppendTo[inputList, s2];
    If[len == 1, nSingleDigits++];
    ]; (* end For[ i ] loop *)
(* if only single digits are present, return this one-row matrix *)
If [nSingleDigits == nInputStrings,
(* the first row has zeros in the positions corresponding to
input strings of length 1. other positions have non-zero entries. *)
firstRow = Table[1, {1}, {iBase}];
For[i = 1, i <= nInputStrings, i++,
If[StringLength[ inputList[[i]] ] == 1,
j = ToExpression[ inputList[[i]] ];
firstRow[[1, j+1]] = 0;
];
];
Return[firstRow]
];
(* now we set up the unique strings that define all the sets. see the schmelzer-baillie paper for details.
example 1: suppose the input string is \{ "314" \}.
let $S$ be the set of all integers not containing this string.
then we make 3 subsets of $S$ :
set 2 is all elements of $S$ that end in "3" ;
set 3 is all elements of $S$ that end in "31" ;
set 1 is all other elements of $S$.
example 2: suppose the input list is \{ "12", "345", "3467" \}.
let $S$ be the set of all integers containing none of these strings.
then we make 4 subsets of S :
set 2 is all elements of $S$ that end in "1" ;
set 3 is all elements of $S$ that end in " 3 " ;
set 4 is all elements of $S$ that end in "34" ;
set 5 is all elements of $S$ that end in " 346 " ;
set 1 is all other elements of $S$.
example 3: complicated example from the schmelzer-baillie paper:

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    suppose the input list is { "0", "2", "4", "6", "8", "55", "13579" }.
    then we make 6 subsets of S:
    set 2 is all elements of S that end in "5" but not "135" ;
    set 3 is all elements of S that end in "1" ;
    set 4 is all elements of S that end in "13" ;
    set 5 is all elements of S that end in "135" ;
    set 6 is all elements of S that end in "1357" ;
    set 1 is all other elements of S.
    *)

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setStrings = { "" }; nSetStrings = 1; (* for first set (first row) *)

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setStrings = { "" }; nSetStrings = 1; (* for first set (first row) *)
For[i = 1, i <= nInputStrings, i++, (* process the ith input string *)
For[i = 1, i <= nInputStrings, i++, (* process the ith input string *)
    s2 = inputList[[i]];
    s2 = inputList[[i]];
    len = StringLength[s2];
    len = StringLength[s2];
    If[len > 1, (* input strings of length 1 are processed elsewhere *)
    If[len > 1, (* input strings of length 1 are processed elsewhere *)
        For[j = 1, j <= len-1, j++,
        For[j = 1, j <= len-1, j++,
            firstJChars = StringTake[s2, j];
            firstJChars = StringTake[s2, j];
                (* make sure that string firstJChars is not already in the setStrings list *)
                (* make sure that string firstJChars is not already in the setStrings list *)
                bFound = False;
                bFound = False;
                For[k = 1, k <= nSetStrings, k++,
                For[k = 1, k <= nSetStrings, k++,
                    If[setStrings[[k]] == firstJChars, bFound = True; Break[] ];
                    If[setStrings[[k]] == firstJChars, bFound = True; Break[] ];
                ];
                ];
                If[bFound == False,
                If[bFound == False,
                    AppendTo[setStrings, firstJChars];
                    AppendTo[setStrings, firstJChars];
                    nSetStrings++;
                    nSetStrings++;
            ];
            ];
        ]; (* end For[ j ] loop *)
        ]; (* end For[ j ] loop *)
    ]; (* end If[len > ... 1] *)
    ]; (* end If[len > ... 1] *)
]; (* end For[ i ] loop *)
]; (* end For[ i ] loop *)
(* at this point, we have setStrings = {1, 3, 34, 346}.
(* at this point, we have setStrings = {1, 3, 34, 346}.
    the number of rows will be the number of elements in setStrings. *)
    the number of rows will be the number of elements in setStrings. *)
nRows = Length[setStrings];
(* now we know the dimensions of the matrix *)
t = Table[1, {nRows}, {iBase}];
(* now set the elements of the matrix *)
For[iRow = 1, iRow <= nRows, iRow++,
    s2 = setStrings[[iRow]];
    (* append each digit to string s2 and see which set the appended string belongs to *)
    For[d = 0, d < iBase, d++,
            (* create a string (s2Appended) that consists of the string
            for this set, with the single digit d appended. *)
        dString = ToString[d];
```

```
s2Appended = StringJoin[s2, dString];
(* if d is a prohibited digit, set t[..] = 0 *)
If[ Length[Position[inputList, dString]] > 0,
    t[[iRow, d+1]] = 0;
    Continue[]; (* next d *)
];
(* if s2Appended equals any of the prohibited input strings, then set t[..] = 0 *)
bFound = False;
For[i = 1, i <= nInputStrings, i++,
    If[ s2Appended == inputList[[i]], t[[iRow, d+1]] = 0;
        bFound = True; Break[] ]; (* end If *)
]; (* end For[ i ] loop *)
If[bFound == True, Continue[] ];
(* if we get here, then s2Appended is not a prohibited input string.
    see if s2Appended contains a prohibited string. *)
For[i = 1, i <= nInputStrings, i++,
    If[ Length[StringPosition[ s2Appended, inputList[[i]] ] ] > 0,
            t[[iRow, d+1]] = 0;
            bFound = True; Break[];
        ];
];
If [bFound == True, Continue[] ];
(* if we get here, then this (row, column) was not set to 0.
    see which set s2Appended belongs in. an example in the schmelzer-baillie
    paper shows that a string ending in "5" could be in two sets ("5" and "135").
    so, first, try to select the set (if any) whose string equals s2Appended.
*)
(* final digits of s2Appended could match more than one setStrings[[i]].
    among the matches, save the value of i for which
    StringLength[ setStrings[[i]] ] is largest.
*)
maxMatched = 0; (* max number of final digits that match *)
iSaved = 0;
For[i = 2, i <= nSetStrings, i++,
    nCompare = StringLength[ setStrings[[i]] ];
    (* if s2Appended is shorter than setStrings[[i]], then it cannot match setStrings[[i]] *)
    If[StringLength[s2Appended] < nCompare, Continue[] ];
    (* compare the last (nCompare) characters of s2Appended and setStrings[[i]] *)
    If[ StringTake[s2Appended, -nCompare] == StringTake[setStrings[[i]], -nCompare],
        bFound = True;
        If[StringLength[ setStrings[[i]] ] > maxMatched,
                maxMatched = StringLength[ setStrings[[i]] ];
                iSaved = i;
```

```
                ];
                ];
        ]; (* end For[ i ] loop *)
        If [bFound, t[[iRow, d+1]] = iSaved];
    ]; (* end For[ d ] loop *)
    ]; (* end For[ iRow ] loop *)
    t (* return this matrix *)
]; (* end of setT *)
kSumX[stringList_?VectorQ, T_?MatrixQ, nDecimals0_Integer, iBase_] :=
Module[
    (* the user calls kSum[ ] or kSumFormatted[ ]. those functions call this
        private function kSumX, which does all the work. *)
    { nDecimals, inputLen, nDecSum, f, c, a, S, z, nInputDigits, totalDigits,
        iMaxBase, goalMult, matB, wTrunc2, i, d, l, m, w, h,
        iPrint = 0,
        xi },
    Clear[A, Psi, extraPol, n, iMaxBase]; (* clear the global variables *)
    (* in base 10, if the input is one string of N digits, then the sum will have N+1 digits.
        if there are additional strings, they can only lower the sum. therefore, the shortest
        string in the list provides an upper bound for the number of digits in the sum. *)
    (* inputLen = Min[StringLength[stringList]]; *)
    (* StringLength[ ] is not Listable in version 4; next line works in versions 4, 5, 6, 7 *)
    inputLen = Min[Map[StringLength, stringList]];
    (* iMaxBase and goalMult depend on the base. their values were determined empirically.
        to insure that we get as many decimal places of precision as the user requested. *)
    (* iMaxBase is roughly the log of 1000 to base (iBase). so, for base 10, iMaxBase = 3.
        for base 2, iMaxBase = 10. for base 100 if we use "Apply[Plus, N[1/S[i, j]^k, nDecSum]]",
        then iMaxBase = 3 uses too much memory.
*)
nDecimals = nDecimals0;
If[nDecimals < 1, nDecimals = 1];
iMaxBase = Ceiling[ Log[iBase, 1000] ];
(* for base 1000, kSum[0, 20, 1000] = 6907.75610104793192687449 .
    for this, we need iMaxBase > 1 *)
If [iMaxBase < 2, iMaxBase = 2]; (* could happen if iBase = 1000 *)
iBaseSave = iBase; (* used in kPartialSum *)
```

```
    (* goalMult = 1.1 is ok for base 100, but is too small for base 10.
    1.4 is usually enough for base 10. an exception: 100th decimal place of kSumFormatted["99", 100]
    is different from that of kSumFormatted["99", 110]. so, use 1.5 for base 10.
    for base 2, we need an even larger multiplier: 3.1 is too small, but 3.2 works.
    for bases between 2 and 10, a goalMult < 3.2 might work, but we will use 3.2.
*)
goalMult = 1.5;
If[iBase < 10, goalMult = 3.2 ];
If[iBase >= 100, goalMult = 1.1 ];
    (* for base 10, the number of correct decimal places turns out to be about (2/3) * nDecSum.
    therefore, adjust nDecSum accordingly, based on the input (nDecimals).
    also, an N-digit string gives rise to a sum with N+1 digits to the left of the
    decimal point. for base 100, the sum has 2N+1 digits to the left of the decimal point.
*)
```

```
nInputDigits = inputLen;
```

nInputDigits = inputLen;
totalDigits = nDecimals + nInputDigits; (* works in base 10 *)
totalDigits = nDecimals + nInputDigits; (* works in base 10 *)
If[iBase > 10, totalDigits = nDecimals + 2*nInputDigits ];
If[iBase > 10, totalDigits = nDecimals + 2*nInputDigits ];
nDecSum = Floor[goalMult * totalDigits];
nDecSum = Floor[goalMult * totalDigits];
(* for large bases (say 1000), we may have to slightly increase the number of digits
(* for large bases (say 1000), we may have to slightly increase the number of digits
to prevent this: kSum[0, 10, 1000] = 6907.756101048 (only 9 decimals shown). *)
to prevent this: kSum[0, 10, 1000] = 6907.756101048 (only 9 decimals shown). *)
If[iBase > 10,
If[iBase > 10,
nDecSum = nDecSum + Ceiling[Log[10, iBase]*nInputDigits]
nDecSum = nDecSum + Ceiling[Log[10, iBase]*nInputDigits]
];
];
(* the computation of A here is the same code as kSumGetA. *)
(* the computation of A here is the same code as kSumGetA. *)
f[j_, l__, m_] := f[j, l,m] = If [T[[l,m]] == j, 1, 0] ;
f[j_, l__, m_] := f[j, l,m] = If [T[[l,m]] == j, 1, 0] ;
n = Dimensions[T] [[1]] ; (* number of rows *)
n = Dimensions[T] [[1]] ; (* number of rows *)
A = (1/iBase) Sum[ Table[f[j, l, m + 1], {j, n}, {l, n}], {m, 0, iBase-1} ] ;
A = (1/iBase) Sum[ Table[f[j, l, m + 1], {j, n}, {l, n}], {m, 0, iBase-1} ] ;
c[k_, w_] := c[k, w] = Binomial[k + w - 1, w] ;
c[k_, w_] := c[k, w] = Binomial[k + w - 1, w] ;
a[k_, w_, m_] := iBase^(-k - w) * c[k, w] * (-1)^w * power2[m, w] ;
a[k_, w_, m_] := iBase^(-k - w) * c[k, w] * (-1)^w * power2[m, w] ;
matB[a_?MatrixQ] := (Inverse[\# - a] - \#) \&[IdentityMatrix[Length[a]]];
matB[a_?MatrixQ] := (Inverse[\# - a] - \#) \&[IdentityMatrix[Length[a]]];
(* function of i, k, and nDecSum *)
(* function of i, k, and nDecSum *)
wTrunc2[i_, k_] := Max[Floor[(nDecSum + 1)/(i - 1) + 1 - k] + 1, 0] ;
wTrunc2[i_, k_] := Max[Floor[(nDecSum + 1)/(i - 1) + 1 - k] + 1, 0] ;
extraPol = nDecSum + 1;
extraPol = nDecSum + 1;
(*
(*
If[iPrint > 0,
If[iPrint > 0,
Print["nDecimals = ", nDecimals, ", iMaxBase = ", iMaxBase, ", goalMult = ", goalMult];

```
    Print["nDecimals = ", nDecimals, ", iMaxBase = ", iMaxBase, ", goalMult = ", goalMult];
```

```
        Print["nDecSum = ", nDecSum];
    ];
*)
```

    S[i_, \(\left.j_{-}\right]:=S[i, j]=\)
    If [i == 1, Complement[Extract[Table[d, \{d, 0, iBase-1\}], Position[T[[1]], j]], \{0\}] ,
    Module[\{ p, el, m, k, elel, mm \},
            \(\mathrm{p}=\) Position[Table[T[[el, m]], \{el, n\}, \{m, iBase\}], j]; h = \{\};
            For \([k=1, k<=\operatorname{Length}[p], k++\),
                \{elel, mm\} \(=\mathrm{p}[\mathrm{k}]] ; \mathrm{h}=\mathrm{Join}[\mathrm{h}, \mathrm{iBase} * \mathrm{~S}[\mathrm{i}-1\), elel] + (mm - 1)]
            ]
            ] ; (* end Module *)
    (* If [iPrint == 1, Print["h[", i, ",", j,"]=", h] ]; *)
h (* return this list of i-digit integers $=\mathrm{S}[\mathrm{i}, \mathrm{j}]$ *)
] ; (* end If *)
(* here are two ways to compute Psi:
$N[A p p l y[P l u s, 1 / S[i, j] \wedge k]$, nDecSum] this is slow but uses little memory
or
Apply [Plus, $N[1 / S[i, j] \wedge k, n D e c S u m]]$ this is fast but uses lots of memory
lst1 = Range[11, 99]
lst1 $=$ Select[Range[11, 99], $\operatorname{Mod}[\#, 10]$ != 0 \&] (* for kSum[0], remove 20, 30, ..., 90 *)
there is a difference in how these two expressions are evaluated:
$\mathrm{N}\left[\right.$ Apply [Plus, $\left.1 / 1 \mathrm{st} 1^{\wedge} \mathrm{k}\right]$, nDecSum]
Apply [Plus, $N[1 / 1$ st1^k, nDecSum] $]$

f2[nDec_] := \{ N[Apply[Plus, 1/lst1], nDec] , Apply[Plus, N[1/lst1, nDec]] \} // N
*)

```
Psi[i_, j_, k_] := Psi[i, j, k] =
    If[i <= iMaxBase,
    (* If [speedMode == 0,
            N[Apply[Plus, 1/S[i, j]^k], nDecSum], (* slow, but uses little memory; N[long rational] *)
            (* Apply[Plus, N[1/S[i, j]^k, nDecSum]] *) (* fast, but uses lots of memory; sum of decimals *)
        (* compute only one 1/S[i, j]^k at a time to save memory; 1/21/2008 *)
        (* xSum = 0;
            For[xi = 1, xi <= Length[S[i, j]], xi++,
                    xSum += N[1/(S[i, j][[xi]])^k, nDecSum]
                ];
                xSum
        *)
        Sum[ N[1 / (S[i, j][[xi]])^k, nDecSum], { xi, Length[S[i, j]] } ] (* 1/24/2008 *)
        ] *) (* end If[speedMode] *)
        Sum[ N[1 / (S[i, j][[xi]])^k, nDecSum], { xi, Length[S[i, j]] } ]
```

```
        (* below, i > iMaxBase *)
        Sum[f[j, l, m + 1]*Sum[a[k, w, m]*Psi[i - 1, l, k + w],
            {w, 0, wTrunc2[i, k]}], {m, 0, iBase-1}, {l, 1, n}]
        ] ; (* end If *)
(*
If[iPrint == -1,
    Print["kSumx: extraPol = ", extraPol, ", n = ", n];
    (* print the Psi values that will be used in computing z, below. the 3rd argument
        is always 1 there, but values of Psi[i-1, j, k] where k > 1 were used above
        to compute Psi[i, j, k]. *)
    Print[" (1) Psi (1..", extraPol, ", 1..", n, ", 1) = ",
            N[Table[Psi[i, j, 1], {i, 1, extraPol}, {j, 1, n}], 5] // TableForm ];
    Print[" (2) Psi (", extraPol, ", 1..", n, ", 1) = ",
        N[Table[Psi[extraPol, j, 1], {j, n}], 5] // TableForm ];
];
*)
    (* use normOne[v] instead of Norm[v, 1] so this can run with Mathematica 4.0 *)
    z = Sum[Psi[i, j, 1], {i, 1, extraPol}, {j, 1, n}] +
        normOne[matB[A] . Table[Psi[extraPol, j, 1], {j, n}] ];
    (* return z, rounded to the number of decimal places requested *)
    If[z == 0, kSumSave = 0, kSumSave = N[z, Floor[ 1 + Log[10, z] ] + nDecimals ] ];
    nDecimalsSave = nDecimals;
(* If[iPrint == 1, Print["z = ", z] ]; *)
    kSumSave (* kSumSave is used in kPartialSum *)
] (* end of kSumX *)
kSum[T_?MatrixQ, nDecimals_Integer] :=
Module[ (* in this version, we input a T matrix, not a string *)
    (* note: the base is not a parameter; the base is the number of columns in the matrix *)
    (* warning - a matrix is a type of list. therefore, this kSum[matrix] function must be
        placed before kSum[list] in this source code file. *)
    { n, iBase, i, stringList = { }, s = ""},
    If[nDecimals < 1, nDecimals = 1];
    n = Dimensions[T][[1]]; (* number of rows *)
    If[Length[ Dimensions[T] ] == 1,
        iBase = Length[T], (* T is a 1 by n matrix *)
        iBase = Dimensions[T][[2]]; (* iBase is the number of columns in the T matrix *)
    ] ;
    (* make a string of length n. it doesn't matter what its digits are; only the length matters. *)
```

```
    For[i = 1, i <= n, i++, s = StringJoin[s, "0"] ];
    AppendTo[stringList, s];
    kSumX[stringList, T, nDecimals, iBase]
] (* end of kSum[T matrix, nDec] *)
kSum[T_?MatrixQ] :=
Module[ (* in this version, we input a T matrix, not a string *)
    (* note: the base is not a parameter; the base is the number of columns in the matrix *)
    { },
    kSum[T, nDefaultDecimals]
] (* end of kSum[T matrix] *)
kSum[inputList_?VectorQ, nDecimals_Integer, iBase_:10] :=
Module[
    { lst, T },
    lst = convertInputListToStrings[inputList];
    If[Length[lst] == 0, Return[0] ];
    T = setT[lst, iBase];
    If[ T[[1, 1]] < 0, Return[0] ];
    kSumX[lst, T, nDecimals, iBase]
] (* end of kSum[list, nDec, base] *)
kSum[inputList_?VectorQ] :=
Module[
    { },
    kSum[inputList, nDefaultDecimals]
] (* end of kSum[list] *)
kSum[s_String, nDecimals_Integer, iBase_:10] :=
Module[
    { stringList = { s } },
    kSum[stringList, nDecimals, iBase]
] (* end of kSum[string, nDec, base] *)
kSum[s_String] :=
Module[
    { stringList = { s } },
    kSum[stringList, nDefaultDecimals]
] (* end of kSum[string] *)
kSum[i_Integer, nDecimals_Integer, iBase_:10] :=
```

```
Module[
    { stringList = { ToString[i] } },
    kSum[stringList, nDecimals, iBase]
] (* end of kSum[integer, nDec, base] *)
kSum[i_Integer] :=
Module[
    { stringList = { ToString[i] } },
    kSum[stringList, nDefaultDecimals]
] (* end of kSum[integer] *)
```

kSumFormatted[T_?MatrixQ, nDecimals_Integer, iBase_:10] :=
Module[ (* in this version, we input a T matrix *)
(* returns a formatted value of type NumberForm, not a number *)
\{ z = kSum[T, nDecimals] \},
$\mathrm{nf}[\mathrm{z}, \mathrm{nDecimals}]$
] (* end of kSumFormatted[matrix, nDec, base] *)
kSumFormatted[T_?MatrixQ] :=
Module[ (* in this version, we input a T matrix *)
(* returns a formatted value of type NumberForm, not a number *)
\{ nDecimals = nDefaultDecimals, $z$ \},
z = kSum[T, nDecimals];
$\mathrm{nf}[\mathrm{z}, \mathrm{nDecimals}]$
] (* end of kSumFormatted[matrix] *)
kSumFormatted[inputList_?VectorQ, nDecimals_Integer, iBase_:10] :=
Module[ (* returns a formatted value of type NumberForm, not a number *)
\{ z = kSum[inputList, nDecimals, iBase] \},
$\mathrm{nf}[z, \mathrm{nDecimals}]$
] (* end of kSumFormatted[list, nDec, base] *)
kSumFormatted[inputList_?VectorQ] :=
Module[ (* returns a formatted value of type NumberForm, not a number *)
\{ nDecimals = nDefaultDecimals, z \},
z = kSum[inputList, nDecimals];
$\mathrm{nf}[\mathrm{z}, \mathrm{nDecimals}]$
] (* end of kSumFormatted[list] *)

```
kSumFormatted[s_String, nDecimals_Integer, iBase_:10] :=
Module[ (* returns a formatted value of type NumberForm, not a number *)
    { stringList = { s } },
    kSumFormatted[stringList, nDecimals, iBase]
```

```
kSumFormatted[s_String] :=
Module[ (* returns a formatted value of type NumberForm, not a number *)
    { stringList = { s } },
    kSumFormatted[stringList, nDefaultDecimals]
] (* end of kSumFormatted[string] *)
kSumFormatted[i_Integer, nDecimals_Integer, iBase_:10] :=
Module[ (* returns a formatted value of type NumberForm, not a number *)
    { stringList = { ToString[i] } },
    kSumFormatted[stringList, nDecimals, iBase]
] (* end of kSumFormatted[integer, nDec, base] *)
```

kSumFormatted[i_Integer] :=
Module[ (* returns a formatted value of type NumberForm, not a number *)
\{ stringList $=$ \{ ToString[i] \} \},
kSumFormatted[stringList, nDefaultDecimals]
] (* end of kSumFormatted[integer] *)
kPartialSum[nDigits_Integer?Positive, nDecimals_Integer] :=
Module[ (* partial sum calculation, using previous kSum *)
(* using the same input just passed to kSum [ ], compute the partial sum of $1 / \mathrm{k}$,
where k has <= nDigits digits (that is, denominators are < $10^{\wedge}$ nDigits).
warning: this could underflow if nDigits $>$ about $2 * 10^{\circ} 9$.
first, call kSum[string, decimals]. then, we can call kPartialSum[digits, decimals].
this function uses the existing values of global variables $A$, extraPol, $n$, Psi,
iBaseSave, and kSumSave, all of which were set in the last call to kSum.
note: if this returns less than the requested number of decimal places, then you must
call kSum again with the desired number of places.
*)
\{ partSum $=0, i, j$, matBTrunc \},
(* formal definition, but this is slow
matBTrunc[a_?MatrixQ, M_] :=
((\# - MatrixPower[a, M+1]).Inverse[\# - a] - \#) \&[ IdentityMatrix[Length[a]] ];
*)
matBTrunc[a_?MatrixQ, M_] :=
Module[
(* instead of computing the exact value of $a^{\wedge}(M+1)$, we first convert "a"
to floating-point, then exponentiate. if enough decimal places are used,
this will produce an accurate value of $\mathrm{a}^{\wedge}(\mathrm{M}+1)$.
*)
\{ nDec, ndigSum, (* a2, *) a3 \},

```
    If[nDecimals < 1, nDecimals = 1];
    ndigSum = 1 + Floor[Log[10, kSumSave]];
    (* nDec = nDecimals + ndigSum + IntegerLength[M+1] + 5; *)
    nDec = nDecimals + ndigSum + (1+Floor[Log[10, M+1]]) + 5; (* for mma version 5.2 *)
    (* a3 = MatrixPower[a, M+1] ; *) (* very slow *)
    (* a2 = iBaseSave * a; a3 = N[MatrixPower[a2, M+1], nDec] / iBaseSave^(M+1); *)
    (* a2 = iBaseSave * a; a3 = MatrixPower[N[a2, nDec], M+1] / iBaseSave^(M+1); *)
    a3 = MatrixPower[N[a, nDec], M+1] ; (* fast *)
    ((# - a3).Inverse[# - a] - #) &[ IdentityMatrix[Length[a]] ]
    ] ; (* end of matBTrunc *)
    If[ (extraPol <= 0) || (kSumSave == 0), (* must call kSum[ ] or kSumFormatted[ ] first *)
        Print["Error: no sum has been computed yet. Call kSum[ ] or kSumFormatted[ ] first."];
        Return[0]
    ] ;
    If[nDigits > extraPol,
        (* Check[ ] forces partSum to be 0 if an overflow or underflow occurs *)
        partSum = Check[ Sum[Psi[i, j, 1], {i, 1, extraPol}, {j, 1, n}] +
            normOne[matBTrunc[A, nDigits - extraPol] . Table[Psi[extraPol, j, 1], {j, n}] ] ,
            (* 0, { General::unfl , General::ovfl } ]; *)
            0, General::unfl ]; (* check only for underflow for mma version 5.2 *)
    ,
    (* else: nDigits <= extraPol; just sum up to i = nDigits *)
    partSum = Sum[Psi[i, j, 1], {i, 1, nDigits}, {j, 1, n}]
    ]; (* end If *)
    If[partSum == 0, 0, N[partSum, Floor[ 1 + Log[10, partSum] ] + nDecimals ] ]
] (* end of kPartialSum[nDigits, nDec] *)
kPartialSum[nDigits_Integer?Positive] :=
Module[ (* partial sum calculation, based on previous kSum *)
(* using the previous input to kSum[ ], compute the
    partial sum of 1/k, where k has <= nDigits digits. *)
    { },
    kPartialSum[nDigits, nDefaultDecimals]
] (* end of kPartialSum[nDigits] *)
Clear[kPartialSumThreshold];
Clear[kPartialSumThresholdX];
kPartialSumThresholdX[pSum0_?NumericQ, nDecDefault_] :=
```

(* this private function is called by public functions kPartialSumThreshold.
using the previous input to kSum [ ], compute the number of digits at which the partial sum exceeds pSum0.
example:
$\mathrm{kSum}[9]=22.920676619264150$.
kPartialSumThreshold[22] tells approximately how far we must go in the series to make the partial sum exceed 22 .
kPartialSumThreshold[22] returns \{30, 31, 21.971055078178619, 22.066017232287173\}. this means that if we add all terms in the series including all those having 30 digits (that is, denominators < 10^30), the sum will be 21.971055078178619, and if we also include terms with denominators having 31 digits, the sum will be 22.066017232287173 . therefore, we would need at least some 31-digit denominators in order to make the sum exceed 22 .

```
in general, the input value (22, in this example) will lie between the partial sums
that are returned:
    21.971055078178619 < 22 < 22.066017232287173.
```

two warnings about using this function:

1. if pSum is very close to the actual sum returned by kSum [ ], you may be asked to call kSum [ ] again, with more decimals, then to call kPartialSumThreshold[ ] again.
2. also, if you specify pSum as a decimal value, mathematica may slightly change your input before kPartialSumThreshold can process it. therefore, if pSum is close to the actual sum, you could get incorrect results.
here is an example showing how this can heppen. suppose you compute $\mathrm{kSum}[314,20]=2299.82978276751833845359$.
then, if you enter
kPartialSumThreshold[2299.8297827675],
this function will print out: Calculate where the partial sum exceeds 2299.8297827675000917225 \{32386, 32387, 2299.82978276750008978870, 2299.82978276750010807399\}
notice that mathematica has changed your input to 2299.829782767500092 . then, because kPartialSumThreshold uses this changed input, the output you got is not the correct output for the input you specified: notice that your input value is not between the two partial sums shown.
there are three ways to prevent mathematica from changing your input.
3. you can use mathematica's backquote notation to specify the accuracy of your input. so, you could enter:
kPartialSumThreshold[2299.8297827675`10]
this time, you get the correct output:
Calculate where the partial sum exceeds 2299.8297827675000000000
\{32381, 32382, 2299.82978276749999808677, 2299.82978276750001646395\}
4. enter your input using quotes:
kPartialSumThreshold["2299.8297827675"]
```
    3. enter an exact rational number:
            kPartialSumThreshold[2299 + 8297827675/10000000000].
    these three methods all produce the correct output for the input you specified.
    the bottom line is this: if you specify a decimal value for the partial sum,
    it is a good idea to use the backslash notation, such as 2299.8297827675``10.
*)
{ i, s, psa, psb, psc, a, b, c, nDec, diff, pSum, iLimit = 30,
    errorReturn = { -1, -1, -1, -1 } },
If[ (extraPol <= 0) || (kSumSave == 0),
    Print["Error: no sum has been computed yet. Call kSum[ ] or kSumFormatted[ ] first."];
    Return[ errorReturn ]
];
pSum = pSum0;
If[pSum <= 0, Return[ errorReturn ] ]; (* error *)
nDec = nDecDefault;
If[nDec < nDecimalsSave, nDec = nDecimalsSave];
If[Accuracy[pSum] < nDec, pSum = SetAccuracy[pSum, nDec] ];
Print["Calculate where the partial sum exceeds ", pSum];
s = kSumSave;
If[pSum > s,
    Print["Value must be less than the actual sum (", s, ")."];
    Return[ errorReturn ]
]; (* error *)
If[pSum >= s,
    Print[pSum, " is too close to the actual sum (", s, ")."];
    Print["Either enter a smaller value, or call kSum again with more than ",
                nDecimalsSave, " decimal places."];
    Return[ errorReturn ]
]; (* error *)
a = 1;
psa = kPartialSum[a, nDec] ;
If[pSum < psa,
    Return[ { 0, 1, 0.0, psa }];
];
b = a;
For[i = 1, i <= iLimit, i++,
    psb = kPartialSum[b, nDec] ;
```

```
    If[psb > pSum, Break[ ] ];
    a = b; psa = psb;
    b = b * 2;
    ];
    (* get here with b such that partial sum > pSum *)
    If[psb == pSum,
        Print[pSum, " is too close to the actual sum."];
        Print["Either use a smaller value, or increase the number of decimals (2nd parameter)."];
        Return[ errorReturn ] (* error *)
    ];
    If[psb < pSum, Return[ errorReturn ] ]; (* error *)
    (* enter the loop below with psa < pSum and psb > pSum.
        compute c = (a + b)/2, then compute psc. *)
    diff = b - a;
    c = a + diff/2;
    For[i = 1, i <= iLimit, i++,
        psc = kPartialSum[c, nDec] ;
        If[psc > pSum,
            b = c; psb = psc;
        ,
        a = c; psa = psc;
        ];
        diff = b - a;
        If[diff < 2, Break[] ];
        c = a + diff/2;
    ]; (* end loop *)
    If[ (pSum <= psa) || (psb <= pSum),
        nDecimalsNeeded = -Floor[Log[10, Abs[psb - psa]]] + 5;
        nDecimalsNeeded = 5*(1 + Floor[(nDecimalsNeeded-1)/5]); (* round up to mult of 5 *)
        If [nDecimalsNeeded > nDec,
            Print["More decimals are needed. You should re-compute kSum to at least ",
                    nDecimalsNeeded, " decimals, then call kPartialSumThreshold again."] ,
            Print["More precision is needed. You should re-compute kSum to more decimals,\
then call kPartialSumThreshold again."];
        ];
        Return[ errorReturn ];
    ];
    Return[ { a, b, psa, psb } ];
]; (* end of kPartialSumThresholdX *)
kPartialSumThreshold[pSumStr_String, nDecDefault_] :=
```

```
Module[
    (* a string was entered. add double backquotes to specify the accuracy,
        then call the private function kPartialSumThresholdX. *)
    { pSum, inputStr2, decPtList, quoteList, nDecimalsInput, nDec2,
        errorReturn ={ -1, -1, -1, -1 }
    },
    decPtList = StringPosition[pSumStr, "."]; (* get list of starting, ending positions *)
    quoteList = StringPosition[pSumStr, "`"];
    If[ (Length[decPtList] == 1) && (Length[quoteList] == 0),
        (* input has one decimal point and no backquotes (usual case); number of
            decimals = number of chars between decimal point and end of string.
            append backquotes and a number to specify the accuracy of the input,
            then convert the new string to a floating-point number with that accuracy.
        *)
        nDecimalsInput = StringLength[pSumStr] - decPtList[[1]][[1]] ;
        nDec2 = Max[nDecimalsInput + 2, nDecDefault];
        inputStr2 = pSumStr <> "`"" <> ToString[nDec2] ;
        pSum = ToExpression[inputStr2];
        (* otherwise (backquotes already in use, or input is an integer):
            just convert the input string to a number *)
        pSum = ToExpression[pSumStr]
    ];
    If[(pSum == $Failed) || (NumericQ[pSum] == False),
        Print["Invalid input"]; (* cannot convert input to a numeric value *)
        Return[ errorReturn ]
    ] ;
    kPartialSumThresholdX[pSum, nDecDefault]
]; (* end of kPartialSumThreshold[string, decimals] *)
kPartialSumThreshold[pSumStr_String] :=
Module[
    { },
    kPartialSumThreshold[pSumStr, nDefaultDecimals]
]; (* end of kPartialSumThreshold[string] *)
kPartialSumThreshold[pSum0_?NumericQ] :=
Module[
(* a numeric value was entered. assume the user has included double backquotes
        if they are needed; this calls the private function kPartialSumThresholdX. *)
    { },
    kPartialSumThresholdX[pSum0, nDefaultDecimals]
];
```

```
kPartialSumThreshold[pSum0_?NumericQ, nDecDefault_] :=
Module[
(* a numeric value was entered. assume the user has included double backquotes
        if they are needed. if they did not, there is no way to include them now.
        this calls the private function kPartialSumThresholdX. *)
    { },
    kPartialSumThresholdX[pSum0, nDecDefault]
];
(* these return the A and T matrices *)
kSumGetT[inputList_?VectorQ, iBase_:10] :=
Module[ (* given the input list, return the matrix A *)
    { lst },
    lst = convertInputListToStrings[inputList];
    If[Length[lst] == 0, Return[{{0}} ] ];
    setT[lst, iBase] (* return this matrix *)
] (* end of kSumGetT[list] *)
kSumGetT[s_String, iBase_:10] :=
Module[ (* given the input string s, return the matrix A *)
    { stringList = { s } },
    kSumGetT[stringList, iBase]
] (* end of kSumGetT[string] *)
kSumGetT[i_Integer, iBase_:10] :=
Module[ (* given the input integer, return the matrix A *)
    { stringList = { ToString[i] } },
    kSumGetT[stringList, iBase]
] (* end of kSumGetT[integer] *)
(* show the current (private context) A matrix for the previously-entered input *)
kSumShowA[] := A;
kSumGetA[inputList_?VectorQ, iBase_:10] :=
Module[ (* given the input list, return the matrix A *)
    { lst2, T, f, j, l, m, A, n },
    (* use only local variables here, not the global variables A and n.
        this allows the user to call kSum, then kSumGetA, then kPartialSum,
        without kSumGetA interfering with the global variables.
    *)
    lst2 = convertInputListToStrings[inputList];
    If[Length[lst2] == 0, Return[ {{0}} ] ];
    T = setT[lst2, iBase];
```

```
    (* the computation of A here is the same code as kSumX *)
    f[j_, l_, m_] := f[j, l, m] = If [T[[l, m]] == j, 1, 0] ;
    n = Dimensions[T][[1]] ; (* number of rows *)
    A = (1/iBase) Sum[ Table[f[j, l, m + 1], {j, n}, {l, n}] , {m, 0, iBase-1} ] ;
    A (* return this matrix *)
] (* end of kSumGetA[list] *)
kSumGetA[s_String, iBase_:10] :=
Module[ (* given the input string s, return the matrix A *)
    { stringList = {} },
    AppendTo[stringList, s];
    kSumGetA[stringList, iBase]
]; (* end of kSumGetA[string] *)
kSumGetA[i_Integer, iBase_:10] :=
Module[ (* given the input integer, return the matrix A *)
    { stringList = {} },
    AppendTo[stringList, ToString[i]];
    kSumGetA[stringList, iBase]
]; (* end of kSumGetA[integer] *)
```

```
kSumTimeAndMemory[nDigits_] :=
```

kSumTimeAndMemory[nDigits_] :=
Module[
Module[
(* display the estimated time and memory needed to calculate kSum[9, nDigits].
(* display the estimated time and memory needed to calculate kSum[9, nDigits].
based on calculations on a dell laptop for 100, 200, ..., 600 digits. *)
based on calculations on a dell laptop for 100, 200, ..., 600 digits. *)
{ nd, t1, t0, tFast, mFast},
{ nd, t1, t0, tFast, mFast},
nd = nDigits;
nd = nDigits;
If[nd < 50, nd = 50];
If[nd < 50, nd = 50];
t1 = Timing[ kSum[9, 100] ][[1]];
t1 = Timing[ kSum[9, 100] ][[1]];
(* to = actual time for kSum[9, 100] on the laptop used for benchmark *)
(* to = actual time for kSum[9, 100] on the laptop used for benchmark *)
t0 = 3.647 ;
t0 = 3.647 ;
tFast = . 55227 + .017407*nd + .00017173*nd^2 + (3.3708*10^-7)*nd^3 ;
tFast = . 55227 + .017407*nd + .00017173*nd^2 + (3.3708*10^-7)*nd^3 ;
mFast = (6.5614*10^6) + 2836.1*nd + 18.432*nd^2 + .0054213*nd^3 ;
mFast = (6.5614*10^6) + 2836.1*nd + 18.432*nd^2 + .0054213*nd^3 ;
(* the following cubic polynomial works better for 100 <= nd <= 1000 *)

```
    (* the following cubic polynomial works better for 100 <= nd <= 1000 *)
```

```
    Print["Estimated time and memory for kSum[9, ", nd, "]:" ];
    Print[Round[tFast*t1/t0], " seconds, ", Round[mFast/1000000], " MBytes of memory"];
    Print[" Note: kSum[99, ", nd, "], kSum[999, ", nd,"] etc., will use more time and memory." ];
]; (* end of kSumTimeAndMemory *)
kSumSetDefaultDecimals[i_Integer] :=
Module[
    (* set the number of default decimal places *)
    { },
    If[i< 1, i = 1];
    If[nDefaultDecimals == i,
        Print["Default number of decimal places is now set to ", i] ,
        Print["Default number of decimal places has been changed from ", nDefaultDecimals, " to ", i]
    ];
    nDefaultDecimals = i;
    nDefaultDecimals (* return this value *)
];
kSumShowDefaultDecimals := nDefaultDecimals;
End[ ] (* end the private context *)
(* protect the exported symbols *)
Protect[kSum, kSumFormatted, kPartialSum, kSumGetT, kSumGetA, kSumShowA];
Protect[kSumTimeAndMemory, kSumSetDefaultDecimals, kSumShowDefaultDecimals];
Protect[kPartialSumThreshold];
EndPackage[ ] (* end the package context *)
```

