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The Compton effect—a classical treatment

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Abstract In spite of the fact that many textbooks proclaim that the photon theory of light is necessary to explain the Compton effect, it is possible to treat it as a classical phenomenon provided the important role of the stationary state is recognised. The classical theory is presented and the quantum rules for the interaction of an electromagnetic wave with a charged particle are established.

Résumé Bien que de nombreux ouvrages prétendent que la description de la lumière en termes de photons est indispensable pour l'interprétation de l'effet Compton, il est cependant possible de donner de ce phénomène un traitement classique, à condition de prendre en compte comme il convient le rôle, important, de l'état stationnaire. L'article présente cette théorie classique et établit les règles quantiques qui commandent l'interaction de l'onde électromagnétique avec une particule chargée.

1. Introduction

I should like to indulge in a little fantasy: to pretend that we stand near the beginning of the 20th century and attempt to discover the laws for the interaction between light and matter using the classical theory of the day, being guided by experiments which, in principle, could be performed near that time. From time to time I shall comment on the conclusions we have reached from the point of view of the present day—such comments will be placed in square brackets.

The view of physics at the time we consider is dominated by the theories of Maxwell, Lorentz and Einstein. We know that Maxwell's equations and the Lorentz force equation explain how electrical charges and currents interact; and we have learnt that light is propagated as a wave of the electromagnetic fields described by these equations. Recently we have learnt the new mechanics of Einstein. He has shown that the transformations required to keep the laws of mechanics invariant between inertial observers are the same as those shown by Lorentz to be necessary to keep the laws of electromagnetism invariant. A synthesis of mechanics and electricity has been achieved. Since the newly discovered fundamental particles of matter, the proton and the electron, are the sources of both mass and charge, we ask ourselves the following: 'How does light (a manifestation of electromagnetic theory) interact with a charged particle (a simple particle of matter)?'

We set out to establish a theory, guided by experiment, of the interaction between a beam of polarised light and a *free* stationary electron. [We note from our present viewpoint in the 1980's that this simple experiment has not been performed—at least to the knowledge of the author. High frequency radiation X-rays and γ-rays have been scattered from electrons bound in atoms. It can then be argued that the electrons are effectively free but three points should be noted. Firstly one observes in the scattered radiation, components of the same frequency as the incident radiation; this part of the scattered radiation is presumably from *bound* electrons. Secondly, the scattering hardly takes place from a *single* electron but rather from a system of electrons confined to a volume less than a cubic wavelength. Thirdly, the electrons in the atom do not have zero momentum as is assumed in the simple theory.]

2. The experiment

The results of experiments have been reported by Compton (1923). [An extensive review and appreciation of the work of A H Compton has been published by Strewer, 1975]. The results show that an incident beam of radiation of frequency f_0 when scattered by an electron to an angle θ in the

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laboratory has a frequency

$$f = f_0[1 + \varepsilon(1 - \cos\theta)]^{-1}. \qquad (1)$$

A series of experiments, performed using beams of various incident frequencies, establishes that the empirical constant ε is proportional to that frequency: $\varepsilon = \tau_e f_0$. Since ε is dimensionless we may decide to call τ_e the Compton period of the electron and determine its value experimentally to be $\tau_e = 8.0933 \times 10^{-21}$ s. Indeed some brilliant experimenter may decide to perform the experiment using other charged particles. [The author believes that an accurate experiment under controlled conditions for any other charged particle has never been done.] He would presumably find that the Compton period is inversely proportional to the mass of the particle and write

$$\varepsilon = k f_0 / m_0 \qquad (2)$$

with the constant k having the measured value of

$$\begin{aligned} k &= \tau_e m_0 \\ &= (8.0933 \times 10^{-21}\,\text{s}) \times (9.1096 \times 10^{-31}\,\text{kg}) \\ &= 7.3726 \times 10^{-51}\,\text{kg s}. \qquad (3) \end{aligned}$$

One perceives perhaps the emergence of a fundamental physical constant. But we must now turn to theory for guidance.

3. The theory

We choose a simple electromagnetic wave, plane and circularly polarised, propagating in vacuum along the z axis. In the laboratory frame this is written

$$\boldsymbol{E}(z, t) = E_0(\hat{\boldsymbol{x}} \cos\phi + \hat{\boldsymbol{y}} \sin\phi)$$

$$\boldsymbol{B}(z, t) = \frac{E_0}{c}(-\hat{\boldsymbol{x}} \sin\phi + \hat{\boldsymbol{y}} \cos\phi)$$

where the phase $\phi = 2\pi f_0(t - z/c)$. The electric amplitude E_0 is that which gives the same irradiance as a linear polarised wave of amplitude $\sqrt{2}\,E_0$, (RMS value E_0). Subscripts zero are used to label all important physical quantities in the incident state before any interaction has occurred. Poynting's vector is constant in time:

$$\boldsymbol{S}_0 = \boldsymbol{E} \times \boldsymbol{B}/\mu_0 = (E_0^2/\mu_0 c)\hat{\boldsymbol{z}}. \qquad (4)$$

Unit vectors of an appropriate cartesian coordinate system are denoted by $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$.

We study the interaction of this wave with an initially stationary particle of charge q and rest mass m_0. Using the Lorentz force equation we establish an equation of motion for the particle. Even in neglect of radiation reaction terms we find it is complicated. However, believing as we do that there is simplicity at the end of all transient behaviour (we recall the simple steady-state of standing waves that persist on a plucked undamped string after the transient complications of the excitation and the reflections have died away), we seek a *steady-state* solution. We note that the rotating electric field will (eventually) guide the particle into a circular path, and the action of the magnetic field on the now moving particle will push it 'down stream'. We therefore seek a steady-state solution in which we have a *constant* down-stream velocity $\boldsymbol{V} = c\beta\hat{\boldsymbol{z}}$ on which is superimposed a *constant* rotation.

We transform to a new frame of reference moving with velocity $\boldsymbol{V}$. We call this the zero-momentum (ZM) frame because, after interaction, the particle has zero linear momentum in it. Quantities measured in this frame are labelled by an asterisk. The Lorentz transformations give us:

$$E^* = E_0\left(\frac{1-\beta}{1+\beta}\right)^{1/2}$$

$$B^* = B_0\left(\frac{1-\beta}{1+\beta}\right)^{1/2}$$

$$f^* = f_0\left(\frac{1-\beta}{1+\beta}\right)^{1/2}. \qquad (5)$$

The phase is invariant:

$$\phi^* = 2\pi f^*(t^* - z^*/c) = \phi.$$

Therefore

$$\left.\begin{aligned} \boldsymbol{E}^*(z^*, t^*) &= E^*(\hat{\boldsymbol{x}} \cos\phi + \hat{\boldsymbol{y}} \sin\phi) \\ \boldsymbol{B}^*(z^*, t^*) &= \frac{E^*}{c}(-\hat{\boldsymbol{x}} \sin\phi + \hat{\boldsymbol{y}} \cos\phi) \\ \boldsymbol{S}^* &= (E^{*2}/\mu_0 c)\hat{\boldsymbol{z}} = \boldsymbol{S}_0\left(\frac{1-\beta}{1+\beta}\right). \end{aligned}\right\} \qquad (6)$$

It is straight-forward to recognise a steady state in this frame; it is illustrated in figure 1(a). The (positively charged) particle is rotating in antiphase to $\boldsymbol{E}^*$ under a centripetal force $\boldsymbol{F}^* = q\boldsymbol{E}^*$. Its velocity $\boldsymbol{v}^*$, tangential to the circular path, is always antiparallel to $\boldsymbol{B}^*$ so there is no magnetic force to further accelerate it downstream. The particle rotates in a circle at a fixed value of z^*, i.e. with a constant recoil velocity $V\hat{\boldsymbol{z}}$ in the laboratory, with an angular velocity $2\pi f^*$ in synchronism with the passing circularly polarised wave of frequency f^*.

The above analysis has presumed that any force of interaction between the magnetic field and a magnetic moment $\boldsymbol{\mu}$ of the particle, $\boldsymbol{F} = \boldsymbol{\nabla}\boldsymbol{B} \cdot \boldsymbol{\mu}$, is zero. It can readily be shown that if $\boldsymbol{\mu}$ has only a z component, this force is indeed zero. Thus for a steady state to be created the spin must be aligned parallel or antiparallel to the orbital motion and the average value (expectation value) of any transverse components must be zero. [We detect here the emergence of rules for the coupling of angular momenta.] The centripetal force equation can be written as

$$4\pi^2 m^* f^{*2} R^* = qE^* \qquad (7)$$

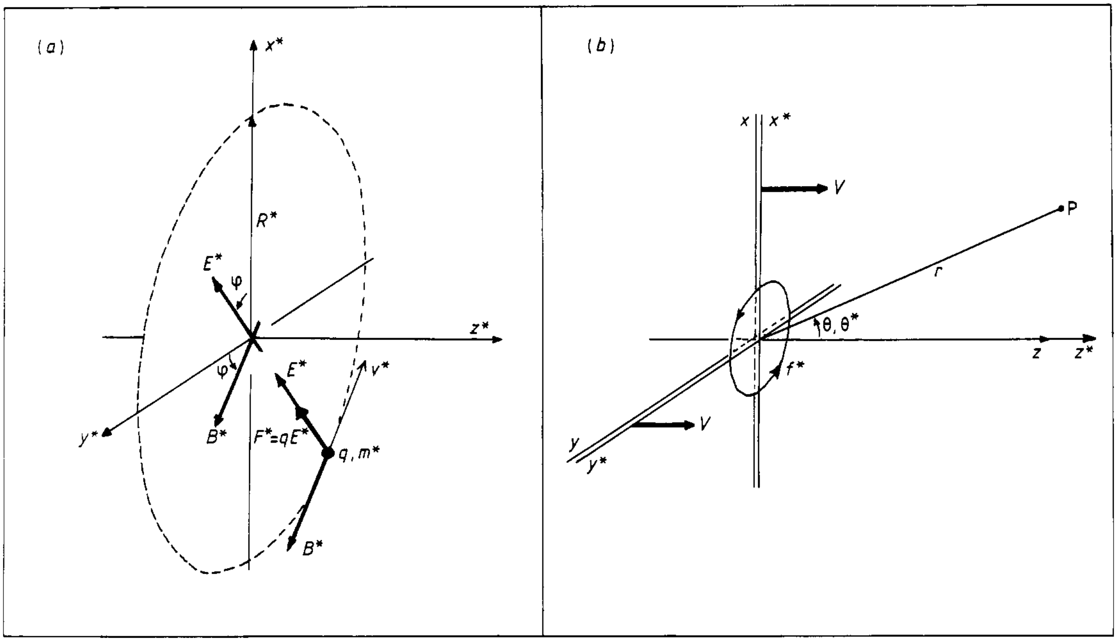

Figure 1(*a*) Motion of the particle (q, m_0) in the zero-momentum (ZM) frame; (*b*), Emission of radiation at an angle θ^* in the ZM frame and θ in the laboratory frame.

where m^* is a relativistic mass in the ZM frame. The equation can be used to give the radius R^* of the steady-state orbit.

The rotating particle radiates (or scatters) an electromagnetic field. What is the frequency observed at a point P at an angle θ in the laboratory (figure 1(*b*)? The frequency of the scattered radiation is f^* in the ZM frame. According to the relativistic Doppler formula the frequency observed in the laboratory is

$$f = f^* \frac{1+\beta \cos \theta^*}{(1-\beta^2)^{1/2}} . \qquad (8)$$

The polar angles in the two frames are related by

$$\cos \theta^* = \frac{\cos \theta - \beta}{1-\beta \cos \theta} .$$

Using this and equation (5) for the relation between the incident frequency and f^*, equation (8) becomes

$$f = f_0 \left[1 + \frac{\beta}{1-\beta} (1-\cos \theta) \right]^{-1} . \qquad (9)$$

This theoretical result for the scattered frequency is to be compared with the empirical result of equation (1); the resemblance is encouraging. We identify the relationship

$$\beta/(1-\beta) = \varepsilon \qquad (10)$$

between the theoretical and experimental parameters and must now seek the meaning of this.

[Some remarks concerning the theory are in order. It will be noted that equation (9) has been established by the use solely of a classical description of radiation. This contrasts markedly with statements frequently made in textbooks that the Compton effect requires a quantum theory of radiation for its explanation. It is indeed true that the usual treatment as originally proposed by Compton established a value for the parameter ε in terms of the incident frequency and physical constants. That theory has however already accepted quantum properties for the energy and momentum of the radiation and has used the conservation laws. As yet we have invoked no such principles. Equation (9) has been established entirely as a result of the Doppler modification of the scattered frequency from a moving resonator. The important principle has been the recognition of the role of a steady state in the intermediate (or interaction) state between the incident and scattered states].

4. Establishing the laws of interaction

So far we have not established what the recoil velocity V is; thus we do not know the coefficient $\beta/(1-\beta)$ in equation (9). To proceed we must make

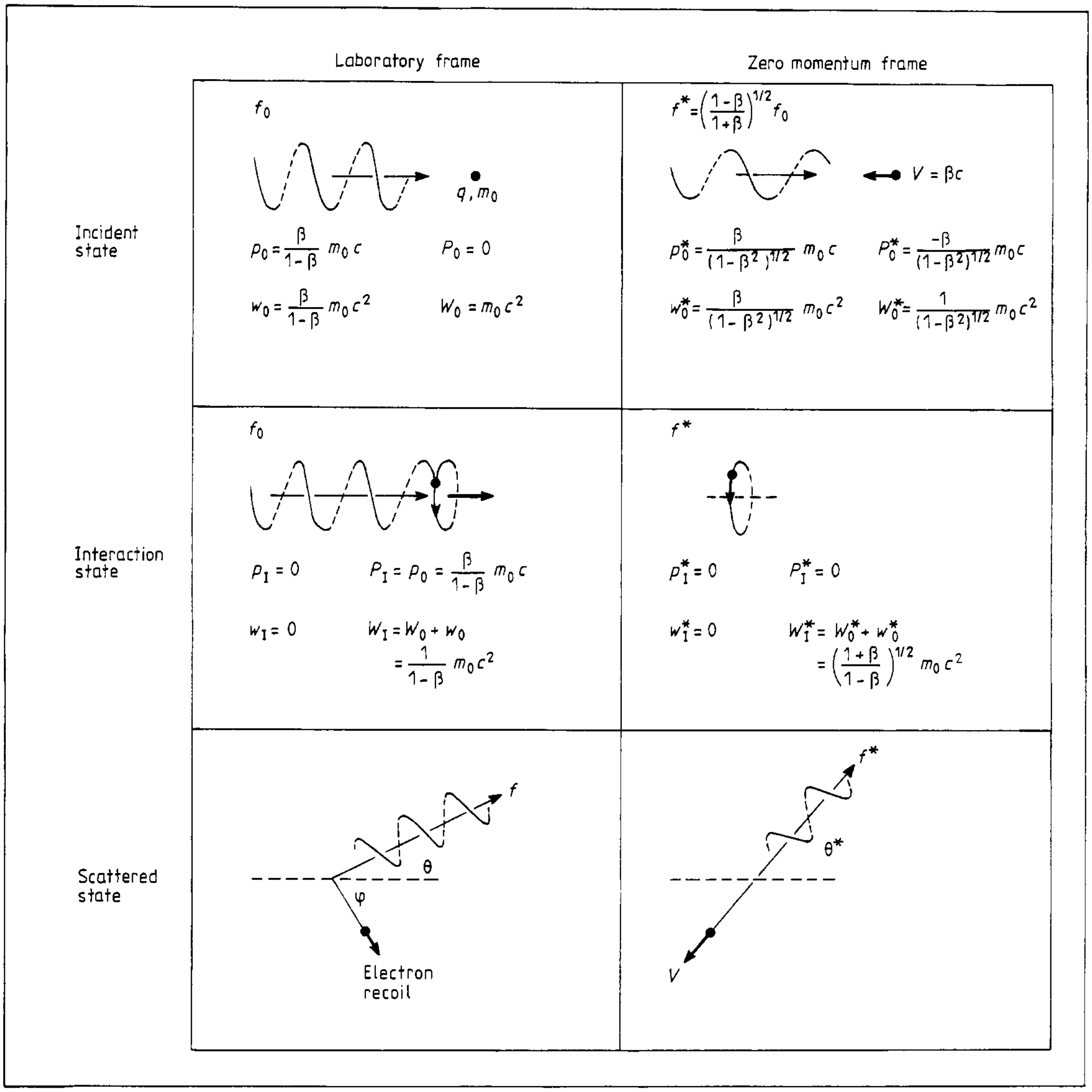

Figure 2 Momenta and energies in the incident, interaction and scattered states observed in the laboratory and ZM frames.

use of the conservation laws of momentum and energy in order to establish the relationship of the state of the system before interaction (incident state) to the excited state of the particle after interaction (interaction state). The linear momentum P and the total energy W of the particle has changed in the interaction; hence a quantity of momentum p and of energy w must have been extracted from the electromagnetic field. The classical theory of radiation based on Maxwell's equations tells us that the energy per unit volume ($u = \varepsilon_0 E_0^2$ in the present case) and the momentum per unit volume ($g = \varepsilon_0 E_0^2/c$) are related by $u = cg$. Hence the amount of energy and momentum extracted from the electromagnetic field (may we introduce the word 'tantum'?†) are related by $w = cp$. The description of momentum and energy interchange is shown in figure 2. The values of momentum and energy in terms of β, m_0 and c are established as follows. We start in the laboratory frame for the incident state where radiation has frequency f_0 and the particle is at rest: $P_0 = 0$, $W_0 = m_0 c^2$. The corresponding values of energy and momentum in the ZM frame are established by Lorentz transforma-

† The author is indebted to Professor G W Series for reminding him of some dimly remembered Latin: 'tantum', so much; 'quantum', how much.

tion: $P_0^* = -\beta m_0 c/(1-\beta^2)^{1/2}$, $W_0^* = m_0c^2/(1-\beta^2)^{1/2}$. Since this is the ZM frame, the field momentum to be transferred must be $p_0^* = -P_0^*$ and the corresponding energy is $w_0^* = cp_0^*$. The values of p_0 and w_0, the 'tanta' of electromagnetic momentum and energy extracted from the field in the laboratory, are then obtained by the reverse Lorentz transformation. All values of momentum and energy in the incident state are established.

After interaction (in the interaction state all important quantities are labelled by a subscript I) p_I^* and P_I^* are both zero. Hence $w_I^* = 0$ and, by conservation of energy $W_I^* = W_0^* + w_0^*$. The corresponding values of the interaction state in the laboratory frame are again established by the reverse Lorentz transformation. The results in figure 2 are self-consistent by Lorentz transformations between frames (horizontally) and conservation laws between states (vertically). We shall discuss the scattered state below.

5. The tantum rules

The results set out in figure 2 show that, in the interaction, a tantum of energy

$$\Delta W = w_0 - w_I = W_I - W_0 = \beta m_0 c^2/(1-\beta) \quad (11)$$

is transferred from the radiation to the particle in the laboratory. Using the identification of equation (10) and the empirical result of equation (2), we get

$$\Delta W = (kc^2) f_0. \quad (12)$$

(The same result is obtained in the ZM frame but with f^* on the right-hand side.) Similarly the tantum of momentum transferred from the radiation to the particle is

$$\Delta P = \beta m_0 c/(1-\beta) = (kc^2)/\lambda_0 \quad (13)$$

where $\lambda_0 = c/f_0$ is the wavelength of the radiation in the laboratory frame. (Again the same result is obtained in the ZM frame with $\lambda^* = c/f^*$ on the right-hand side).

We also notice that angular momentum is transferred to the particle. The author does not know of a properly covariant description for angular momentum but in the appendix justifies the formula

$$L = \frac{W^2 - c^2P^2 - m_0^2c^4}{4\pi f(W - cP)} \quad (14)$$

as appropriate to the present situation, where W, P and f are the total energy, linear momentum and driving frequency respectively in the frame being used. In the incident state, in both the laboratory and the ZM frames, $W^2 - c^2P^2 = m_0^2c^4$ and the angular momentum is zero. In the interaction state, substitution of values of W and P from figure 2 together with the appropriate value of f (f_0 for the laboratory and f^* for the ZM frames) leads to the result that, in either frame, $L = (kc^2)/2\pi$. We deduce that the transfer of angular momentum is

$$\Delta L = (kc^2)/2\pi. \quad (15)$$

The quantity kc^2 occurs in each of the tantum transfer equations (12), (13) and (15). It would appear to be a fundamental physical constant deserving a symbol of its own (h). Using the previously measured value of k in equation (3) we write

$$h = kc^2$$
$$= (7.3726\times10^{-51}\ \mathrm{kg\,s})\times(2.99793\times10^{8}\ \mathrm{m\,s^{-1}})^2$$
$$= 6.6262\times10^{-34}\ \mathrm{J\,s}.$$

From our standpoint at the beginning of the century we recognise our tantum of energy to have the same property as the 'quantum' of energy introduced by Planck (1901). We refer to his nomenclature and write the *quantum rules for interaction:*

> when electromagnetic radiation of frequency f interacts with a charge–matter system it causes a change of energy of hf, a change of linear momentum of h/λ, and a change of angular momentum of $h/2\pi$, where Planck's constant $h = 6.6262\times10^{-34}$ J s.

Furthermore, in establishing these rules, we have established a relationship between the theoretical and empirical parameters (β and ε) and the incident frequency f_0: from equations (11) and (12),

$$\varepsilon = \frac{\beta}{1-\beta} = hf_0/m_0c^2$$

or

$$\beta = \frac{\varepsilon}{1+\varepsilon} = hf_0/(hf_0 + m_0c^2). \quad (16)$$

Thus the equation for Compton scattering can be written as

$$f = f_0\left[1 + \frac{hf_0}{m_0c^2}(1-\cos\theta)\right]^{-1} \quad (17)$$

or in terms of wavelength,

$$\lambda = \lambda_0 + \frac{h}{m_0c}(1-\cos\theta). \quad (18)$$

[The result is identical to that achieved by Compton (1923) using a photon concept and regarding the scattering as an elastic collision between corpuscles. It is of interest to note that Compton wrote '.... if an X-ray were scattered by an electron moving in the direction of propagation at a velocity βc, the frequency of the ray scattered at an angle θ is given by the Doppler principle as

$$f = f_0\left[1 + \frac{\beta}{1-\beta}(1-\cos\theta)\right]^{-1}.$$

It will be seen that this is of exactly the same form as

$$f = f_0[1 + \varepsilon(1 - \cos\theta)]^{-1}$$

derived on the hypothesis of the recoil of the scattered electron. Indeed if $\varepsilon = \beta/(1-\beta)$ or $\beta = \varepsilon/(1+\varepsilon)$, the two expressions become identical. It is clear, therefore, that so far as the effect on the wavelength is concerned, we may replace the recoiling electron by a scattering electron moving in the direction of the incident beam at a velocity such that $\beta = \varepsilon/(1+\varepsilon)$. We shall call βc the 'effective velocity' of the scattering electrons'.

Compton, however, gave no justification for the relation between β and ε, nor did he develop a model which gives an explanation of the scattering phenomenon in terms of β.

In contrast to this completely corpuscular model of Compton, Schrödinger (1927) developed a completely wave-like model in which the electromagnetic waves undergo Bragg reflection from a set of standing matter-waves produced by the superposition of the de Broglie waves for the incident and scattered electron. As Schrödinger points out it is necessary to regard this wave scattering as a steady-state process. It is interesting therefore that the present paper has used more commonly accepted models (at least from the classical point of view), a wave for the radiation and a corpuscle for the electron.]

6. The scattered state

It is necessary to consider the state of the system after the scattering of the radiation has taken place. Since our model has shown that between the incident and the interaction states (figure 2) an amount of energy hf_0 and an amount of momentum h/λ_0 has been absorbed from the electromagnetic radiation (hf^* and h/λ^* respectively in the ZM frame), we must presume that a similar interchange occurs when the scattered field is created by an emission process. This is illustrated in the ZM frame of figure 2, giving an opposite momentum for the electron. When this is transformed to the laboratory frame one obtains the frequency f as a function of θ (equation (9)) and an expression for the momentum of the electron and the angle of recoil

$$\cos\phi = (1+\varepsilon)\tan\theta/2.$$

One is thus led to the view that the observation of scattered radiation in a direction θ is *always* accompanied (within the indeterminacies caused by the finite apertures of the measuring apparatus and the lack of precise knowledge of the position of the scattering electron) by a unique recoil momentum. This model then implies that the radiation itself exists in quanta of defined energy, momentum and angular momentum. The extreme view is that radiation exists as corpuscles, rather remote from the wave-model with which we started. The stage is set for many years of discussion about the 'real nature of light'. The author takes the view that the only 'real' things are the observations of measurable events in detectors. The nature of light itself is unknowable. The important thing is to have a model which will enable the results of experiments to be predicted. From this point of view a wave model with quantum rules for interaction may be just as successful as a corpuscular model which incorporates wave-like properties in order to explain interference phenomena.

It is also possible at this point to establish expressions for the intensity and the polarisation of the scattered light and for the scattering cross section. One starts by using dipole theory in the ZM frame (charge q, rotating at radius R^* at frequency f^*—see equation (7)) and making a Lorentz transformation to the laboratory frame. The result, which agrees as might be expected with that deduced by Compton (1923), is not acceptable however. As shown by Klein and Nishina (1929) a more exact result is obtained by taking into account the spin properties of the electrons using Dirac's relativistic equation for electron motion. [It is interesting to note that Klein and Nishina's treatment is semi-classical in its presentation. The treatment given by Tamm (1930) and presented by Heitler (1953) is quantum electrodynamical.]

Appendix

Justification for equation (14).

For linear motion, the invariant equation relating the linear momentum and total energy of a *free* particle is written

$$-P^2 + \frac{W^2}{c^2} = m_0^2c^2 \quad (A.1)$$

or, in the notation of covariant 4-vectors,

$$P^\mu P_\mu = m_0^2c^2.$$

Equation (A.1) can be written as

$$\frac{P^2}{2m} = \frac{W^2 - m_0^2c^4}{2W} \quad (A.2)$$

where $m = W/c^2$ is the relativistic mass. The right-hand side is non-zero only when the system has some form of dynamical energy (i.e. other than rest-mass energy). The left-hand side, $P^2/2m$, tends to the kinetic energy for a free particle in the low velocity limit. An equivalent expression for a rotating particle would be $L^2/2I$ where L is the angular moment and I the moment of inertia. This also tends to the kinetic energy of an orbiting particle in the non-relativistic limit. However it must be remembered that such a particle is not free but bound by forces to the centre of rotation and has potential energy as well as kinetic energy. The *total* dynamical energy will be greater than $L^2/2I$. In the present case the potential energy is proportional to the square of the displacement from the centre of rotation (from the integration of equation (7) from the centre to the final radius). The virial theorem gives the result that the potential energy equals the kinetic energy; the total dynamical energy is twice the kinetic energy. Therefore if the left-hand side is to represent the total *dynamical* energy it must be L^2/I. Writing $I = L/\omega$ we obtain

$$L = \frac{W^2 - m_0^2c^4}{2\omega W}. \quad (A.3)$$

An invariant form of this involving covariant 4-vectors can be written

$$L = \frac{P^\mu P_\mu - m_0^2c^2}{2k^\mu P_\mu} \quad (A.4)$$

where $P^\mu = (\boldsymbol{P}, W/c)$ is the 4-momentum of the particle and $k^\mu = (\boldsymbol{k}, \omega/c)$ is the 4-vector of the wave propagation with $|\boldsymbol{k}| = \omega/c$. This electromagnetic wave provides the driving frequency.

Applied to the ZM frame after interaction, equation (A.4) reduces to equation (A.3) with $W = W_I^*$ and $\omega = 2\pi f^*$. Applied to the laboratory frame after interaction it reduces to equation (14) with $W = W_I$, $P = P_I$, $\omega = 2\pi f_0$. In each case L is evaluated to be $h/2\pi$. The left-hand side of equation (A.4) is indeed an invariant scalar.

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