The Emissivity of Transparent Materials

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The emission of thermal radiation by transparent materials is reviewed from first principles and is compared with the more familiar emission of radiation by opaque surfaces. The comparison leads to an expression for volume emissive power, which is an important concept for discussions of radiative effects in glass. The present treatment differs from that of McMahon in that it takes account of the diffuse character of radiation. As a result, it also constitutes a simple proof of the often overlooked fact that the radiant flux within a transparent radiator exceeds that emitted into air by a factor approximately equal to the square of the refractive index. Using these concepts, the spectral emissivities of isothermal transparent sheets are expressed in terms of their thickness and the optical properties of their materials. The results are illustrated by a discussion of the total hemispherical emissivities of sheets of window glass at various temperatures. The commonly accepted value of about 0.91 is the same for all glasses having a refractive index of 1.5. However, it applies only for sheets above a certain minimum thickness. For window glass this ranges from 3/16 in. at 200°C. to as much as 8 in. at 1000°C. At 1000°C. a sheet 3/16 in. thick has an emissivity of only 0.59. The application of results to calculations of the radiative cooling of transparent sheets is briefly indicated.

I. Introduction

THE EMISSION of radiation by transparent materials is a subject of fundamental importance to the glass industry. To cite but two examples, the application of optical pyrometry to the measurement of glass temperatures depends on it, and the rate of cooling of glassware is in no small part determined by it. Yet the subject has, until recently, received relatively little attention.

In the more familiar problems of heat transfer by radiation one deals with radiant exchanges between opaque bodies that are separated by transparent media.* Insofar as one is aware of the latter, they are usually very transparent, as for example air, or in the form of relatively thin sheets such as windows; and one is more concerned with the radiation they transmit than with any radiation they may emit. One tends, almost, to forget that they are not *perfectly transparent*. Yet, insofar as they are not perfectly transparent, they absorb

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* The term "transparent" is here used in a broad sense, not restricted to visible radiation. Insofar as this study deals with heat transfer, and therefore with infrared radiation rather than with visible radiation, the term "diathermanous" would be more appropriate.

radiation and must therefore also be capable of emitting radiation. Thus *moderately transparent* media differ from most opaque ones in degree rather than in kind.

As a result of greater familiarity with opaque materials, the emission of radiation is commonly regarded as a surface phenomenon. In fact, however—and this is fundamental to the treatment of transparent radiators—it is a bulk phenomenon. One purpose of this paper is to elaborate the parallelism between this view and the familiar description of opaque-solid radiation in terms of a surface emissive power given by the Stefan-Boltzmann law and an angular distribution approximated by Lambert's cosine law. A study of the emission of radiation by moderately transparent materials would thus seem to be a worth-while endeavor not only because of the technical importance of the subject, but also because it may lead to a better understanding of the emission of radiation in general.

(1) Review of Related Work

A valuable contribution to the study of transparent materials was made by McMahon,[1] who derived an equation analogous to Kirchhoff's law, which in its usual form applies to opaque materials only. McMahon's generalized form of this law relates the spectral emissivity of a plane-parallel sheet of a moderately transparent material to its reflectivity and transmissivity, also taking into account multiple internal reflections within the sheet. In the course of his derivation McMahon introduced the concept of the *spectral volume emissive power* of materials. This is a parameter of fundamental importance in discussing the emission of radiation within transparent bodies and will be shown to underlie a clearer understanding of the already familiar concept of the emissive power of opaque surfaces.

Although McMahon's final results are correct, his treatment is incomplete in two respects. One is that it is concerned with unidirectional radiation only. As such it was satisfactory for his purposes, which were to correlate emission and transmission measurements made on collimated beams normal to sheets of glass.[2] However, in problems of radiant heat transfer, it is the hemispherical rather than the normal emissivity that is of interest. Secondly, the treatment fails to show the dependence of volume emissive power on the refractive index of the material.

Kellett[3] discussed the mechanism by which thermal radiation between successive layers of a moderately transparent material can augment heat transfer by true thermal conduction. His analysis of this "radiative conduction" was based on McMahon's expression for volume emissive power. It was later shown to be incomplete, partly because this expression was itself incomplete and partly because Kellett's treatment, like McMahon's, was restricted to unidirectional radiation. Czerny and Genzel[4] have, since then, shown the importance of treating radiant heat transfer as a three-dimensional phenomenon even where there is a net transfer of energy in one dimension only. Kellett subsequently amended his earlier paper without, however, giving a proof of his correction.[5] Indeed, McMahon's treatment of volume emissive power appears, to date, to have remained unamended. Geffcken,[6] who treated the same problem as Kellett, did so without employing the concept of volume emissive power. He proceeded from a statement that the rate of emission of radiation by a black-body radiator into a mass of glass is n^2 times that given by the familiar Stefan-Boltzmann law, n being the refractive index of the glass. This may seem surprising, for, accustomed as one is to perceiving radiation in air, one has come to regard the Stefan-Boltzmann constant as something absolute. In fact, however, the dependence of the rate of emission on the refractive index of the medium surrounding the radiator was discussed by Drude[7] as early as 1912. In spite of this, and for all its significant bearing on any discussion of radiant heat transfer in glass, it does not seem to have been recognized by glass technologists in this country.

To sum up, it is seen that the enquiry into what goes on within moderately transparent radiating media was not undertaken in any very systematic manner. McMahon and Kellett started from first principles but did not allow for the three-dimensional character of radiation and thus failed to note the importance of the refractive index. The German workers took account of both of these but, aiming directly for the radiative conductivity, by-passed a discussion of volume emissive power.

(2) Object of This Paper

Against this background, it is the purpose of this paper to treat the emission of radiation by transparent materials from first principles, thereby linking existing results and laying a foundation for further work. More specifically, the paper sets out to do three things: (1) to demonstrate anew, and from an engineer's practical point of view, the influence of the refractive index on the rate of radiant emission, (2) to apply these findings to a discussion of the rates of emission from thin sheets of glass, and (3) to touch upon the radiative redistribution of energy *within* sheets of glass.

II. Determination of Volume Emissive Power

The starting point of these considerations is the fact that at elevated temperatures all matter radiates, regardless of its position in space. In other words, the emission of radiation is essentially a bulk phenomenon. That this is not more generally recognized is probably due to the fact that, in the more familiar case of opaque materials, none of the radiation originating in the interior of a body reaches its surface; and all the radiation that does leave the radiator must, by definition, have originated at its surface. On the other hand, a part of the radiation originating in the interior of a moderately transparent material does reach the surface and cross into the medium surrounding the radiator. It is the entry of a radiant flux into the medium surrounding the radiator (usually air) that is commonly perceived as *emission.* This must be contrasted with what might be called the *primary emission* of radiation from elemental particles of matter, which goes on in the interior of bodies regardless of whether its effects reach the outside.

The fact that some of the radiation emerging from the surface of a transparent body originates from its interior has some consequences that are immediately evident. For example, if a body of glass is cooling, the relatively low rate of primary emission from the cool surface zone may be aug-

[1] H. O. McMahon, "Thermal Radiation from Partially Transparent Reflecting Bodies," *J. Opt. Soc. Amer.*, **40**, 376–80 (1950).

[2] H. O. McMahon, "Thermal Radiation Characteristics of Some Glasses," *J. Am. Ceram. Soc.*, **34** [3] 91–96 (1951).

[3] (*a*) B. S. Kellett, "Steady Flow of Heat Through Hot Glass," *J. Opt. Soc. Amer.*, **42** [5] 339–43 (1952); *Ceram. Abstr.*, **1953**, March, p. 41*h*.
(*b*) B. S. Kellett, "Transmission of Radiation Through Glass in Tank Furnaces," *J. Soc. Glass Technol.*, **36** [169] 115–23T (1952); *Ceram. Abstr.*, **1952**, November, p. 201*f*.

[4] Marianus Czerny and Ludwig Genzel, "Über die Eindringtiefe räumlich diffuser Strahlung in Glas" (Depth to Which Diffuse Radiation Penetrates Glass), *Glastech. Ber.*, **25** [5] 134–39 (1952); *Ceram. Abstr.*, **1953**, July, p. 117*h*.

[5] B. S. Kellett, correction to footnote 3(*b*), *J. Soc. Glass Technol.*, **37** [178] 268 (1953).

[6] Walter Geffcken, "Zur Fortleitung der Wärme in Glas bei hohen Temperaturen, I," (Transmission of Heat in Glass at High Temperatures, I), *Glastech. Ber.*, **25** [12] 392–96 (1952); *Ceram. Abstr.*, **1954**, February, p. 31*h*.

[7] P. Drude, Lehrbuch der Optik, p. 494. S. Hirzel, Leipzig, 1912.

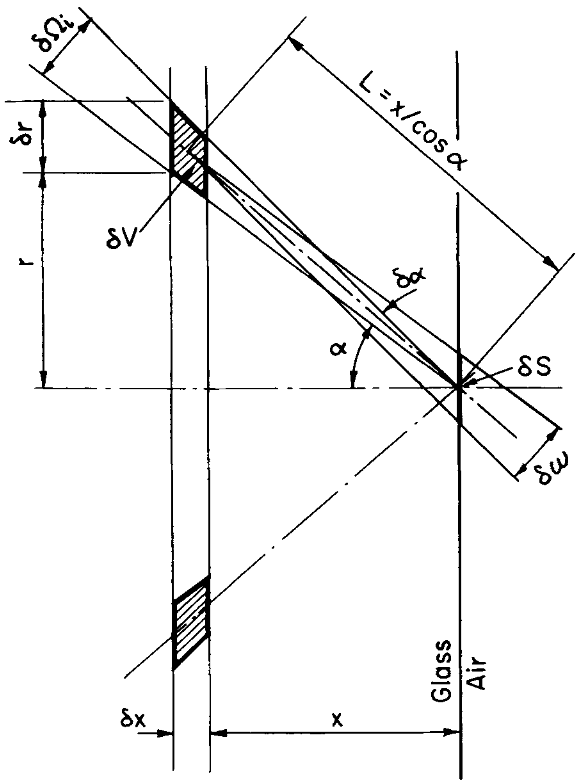

Fig. 1. Primary radiation reaching the surface of a semi-infinite solid.

mented by radiation originating from the hotter interior, which makes the interpretation of readings of a radiation pyrometer somewhat of a problem. However, leaving temperature gradients out of the picture for the present, we return to the basic fact that the emission of primary radiation is a bulk phenomenon, which is also temperature dependent.

To give this concept quantitative expression, we define, as McMahon did, a *spectral volume emissive power* j_λ. This is a measure of the monochromatic power radiated at a given temperature by a unit volume of material into a unit solid angle of space. The reason for speaking of the *spectral* volume emissive power is that, in general, the power of a material to radiate will differ for radiation of different wave lengths. The "per unit solid angle" is brought in as a reminder that radiation is a diffuse process which, in an isotropic material such as glass, proceeds uniformly in all directions.

(1) Primary Radiation Within a Semi-Infinite Body of Transparent Material

We proceed to use this concept to determine what radiation reaches the surface of an isothermal semi-infinite body of a transparent material *from within*. To illustrate the physical mechanisms involved, an outline of the mathematical treatment is given.*

Consider a semi-infinite solid of a transparent material at a uniform temperature T, at which its spectral absorption coefficient is γ_λ cm.$^{-1}$ and its spectral volume emissive power is j_λ cal. per cm.^{3} · sterad.·sec.·μ. Referring to Fig. 1, consider further the primary radiation that reaches a small element δS of the surface of the solid from various elements of volume, such as δV, in the interior of the solid.

Consider δV to be part of a ring of radius r, radial thickness δr, and axial thickness δx. At the center of the ring the small volume δV subtends an angle $\delta\varphi$. Thus, using cylindrical polar coordinates, the elemental volume is

$$\delta V = r\,\delta\varphi\,\delta r\,\delta x$$

or, in terms of the coordinates x, φ, and α

$$\delta V = x^2 \frac{\sin\alpha}{\cos^3\alpha}\,\delta x\,\delta\varphi\,\delta\alpha$$

Let $\delta\omega$ and $\delta\Omega_i$ represent the solid angles subtended, respectively, by δS at δV and by δV at δS. These are given by

$$\delta\omega = \delta S\,(\cos\alpha)/L^2 = \delta S\,(\cos^3\alpha)/x^2$$

and

$$\delta\Omega_i = r\,\delta\varphi\,\delta r\,(\cos\alpha)/L^2 = r\,\delta\varphi\,\delta r\,(\cos^3\alpha)/x^2$$

At a wave length λ, and per unit interval of wave length, the element of volume δV emits radiation at the (spectral) rate

$$j_\lambda\,\delta V \qquad [\text{cal./sterad.}\cdot\text{sec.}\cdot\mu]$$

of which

$$j_\lambda\,\delta V\,\delta\omega \qquad [\text{cal./sec.}\cdot\mu]$$

is directed toward the small element of area δS. The spectral rate of arrival of energy at δS from δV is less than $j_\lambda\,\delta V\,\delta\omega$ because of the absorption of radiation along the path of length $L = x/\cos\alpha$. It is given by

$$\delta Q_\lambda = j_\lambda\,\delta V\,\delta\omega\,e^{-\gamma_\lambda x/\cos\alpha} \qquad [\text{cal./sec.}\cdot\mu]$$

The corresponding elemental contribution to the spectral flux $W_{\lambda i}$ reaching the surface from inside the solid is

$$\delta W_{\lambda i} = \delta Q_\lambda/\delta S \qquad [\text{cal./cm.}^2\cdot\text{sec.}\cdot\mu]$$

and, as this arrives at δS within a solid angle $\delta\Omega_i$, the corresponding elemental spectral intensity is

$$\delta I_{\lambda\alpha} = \delta Q_\lambda/\delta S\,\delta\Omega_i \qquad [\text{cal./cm.}^2\cdot\text{sterad.}\cdot\text{sec.}\cdot\mu]$$

The present interest is in the internal radiant intensity and flux at the surface of a semi-infinite solid. Substituting for δQ_λ, δS, and $\delta\Omega_i$, these are obtained as

$$I_{\lambda\alpha\infty} = \int_{x=0}^{x=\infty} \frac{dQ_\lambda}{dS\,d\Omega_i}$$

$$= \int_0^\infty j_\lambda e^{-\gamma_\lambda x/\cos\alpha}\,dx \qquad (1)$$

$$= (j_\lambda/\gamma_\lambda)\cos\alpha \qquad [\text{cal./cm.}^2\cdot\text{sterad.}\cdot\text{sec.}\cdot\mu] \qquad (2)$$

and

$$W_{\lambda i\infty} = \int_V \frac{dQ_\lambda}{dS}$$

$$= 2\pi\int_0^{\pi/2}\int_0^\infty j_\lambda \sin\alpha\,e^{-\gamma_\lambda x/\cos\alpha}\,dx\,d\alpha \qquad (3)$$

$$= \pi j_\lambda/\gamma_\lambda \qquad [\text{cal./cm.}^2\cdot\text{sec.}\cdot\mu] \qquad (4)$$

Equation (2) shows that the intensity of primary radiation reaching the surface is proportional to the cosine of the angle of incidence. This is Lambert's cosine law, which is usually regarded as empirical. Here it has been proved as a necessary consequence of the assumptions made, namely, that the radiating body is of great thickness, that elemental volumes radiate uniformly in all directions, and that the attenuation of radiation obeys the well-known exponential relation.

It might be noted that the foregoing expressions are in terms of the as yet undetermined spectral volume emissive power j_λ.

(2) Radiation Emitted by a Semi-Infinite Body of Transparent Material

The radiation reaching the surface from the interior is not that emitted, for at the surface refraction and internal re-

* A list of symbols is given on p. 287.

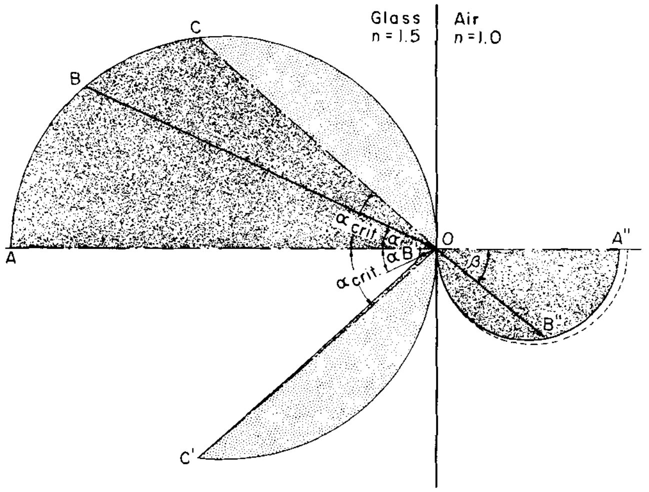

Fig. 2. Polar diagram of radiant intensities at the surface of a semi-infinite transparent radiator.

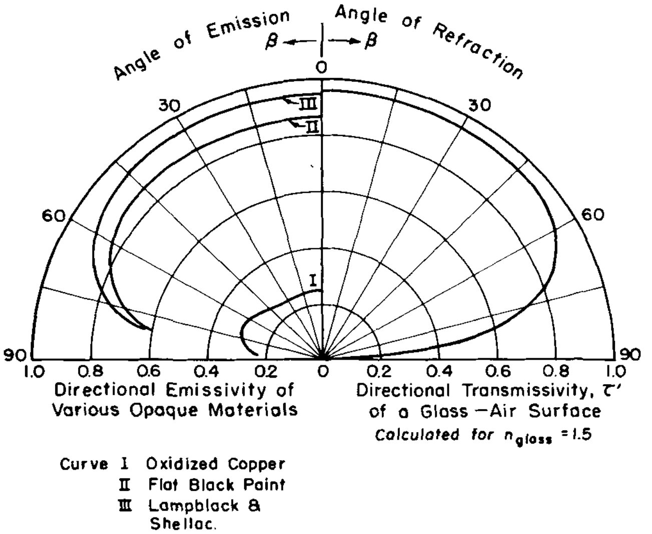

Fig. 3. Angular variation of the transmissivity of a glass-air surface and the emissivities of various opaque materials. Left half of figure after Umur, et al., footnote 8(b).

flection take place. The intensity relationships involved in these processes are briefly treated in Appendix I (see p. 285).

Figure 2 illustrates the results one obtains. It is a polar diagram of the intensities of radiation on both sides of the surface of a semi-infinite radiator of a transparent material. On the left side are shown the beams of primary radiation converging upon a point in the surface from the interior of the radiator. As was shown above, their intensities vary with the cosines of their angles of incidence, which makes their polar diagram a circle. One half of this is shown. At the surface the laws of reflection and refraction take over. The radiation incident on the surface at angles greater than the critical angle, α_{crit}, is totally internally reflected. Of the remainder some is internally reflected, but the bulk is transmitted to emerge as the "emitted radiation" Upon emergence, the transmitted part of the radiation, which was originally contained in a cone of semi-angle equal to the critical angle, is spread out to fill the entire half-space above the surface. Its intensity, which is power per unit solid angle, is correspondingly reduced. It is shown to be

$$I_{\lambda\beta\infty} = \frac{j_\lambda}{\gamma_\lambda} \frac{\tau'}{n^2} \cos\beta \tag{5}$$

In the foregoing expression β is the angle of refraction or "emission," as given by Snell's law

$$\sin\beta = n \sin\alpha$$

and τ' is the directional transmissivity of the surface, i.e. the fraction of the energy of the incident primary beam that is refracted across the surface.* Thus, while the intensity of primary radiation reaching the surface from within a semi-infinite solid ($I_{\lambda\alpha\infty}$) was seen to obey Lambert's cosine law exactly, the angular variation of $I_{\lambda\beta\infty}$, the intensity of emitted radiation, deviates from the simple cosine law to the extent of the variation of τ' with angle of emission β. This variation, for a glass having a refractive index of 1.5, is shown in polar coordinates in the right half of Fig. 3. τ' is seen to be practically constant for angles of emission up to about 55° and to decrease progressively more rapidly at larger values of β. Referring again to equation (5), this means that for the greater part of the radiation emitted by a semi-infinite transparent radiator the radiant intensity in any direction is proportional to the cosine of the angle of emission, and that departures from this simple cosine relation become significant only at relatively large values of the angle of emission. In this respect the radiation from semi-infinite transparent radiators resembles that from opaque nonmetallic radiators, for which similar departures from Lambert's cosine law have long been known (see left half of Fig. 3 and references to Ribaud and Brun and to Umur *et al.*[8]).

This similarity suggests that the radiant flux emitted *across* the surface of a semi-infinite transparent radiator may be identical with that usually regarded as being emitted *by* the surface of an opaque radiator. A close parallel might indeed be expected, for the present considerations, although concerned with transparent materials, have thus far been restricted to infinitely thick bodies, which are effectively opaque in that no radiation can pass through them. (The transmissivity τ' is that of the surface, i.e. the complement of surface reflectivity, and not the transmissivity of the entire body.) It will be shown that, in the present context, the term "infinite thickness" is a relative one, meaning a thickness so great that additional thickness does not noticeably affect the property under study. Thus the magnitude of the "infinite thickness" required to make a slab of a transparent material opaque depends on the degree of transparency of the material; it will be large if the material is very transparent, small if the material is only moderately transparent or relatively opaque, and vanishingly small if the material is intrinsically opaque. Thus the observed applicability to opaque radiators of the "modified cosine law" (equation (5)), which was derived for semi-infinite transparent radiators only, may be explained by the hypothesis that radiation conditions obtaining in a great thickness of a transparent material are reproduced in relatively opaque materials on a very much smaller scale of depth.

(3) Determination of Volume Emissive Power

This hypothesis is next used to express the postulated volume emissive power in terms of the familiar emissive power of surfaces. It is done (see Appendix II) by equating the calcu-

* The prime is used to indicate that τ' is a function of α and therefore also of β.

[8] (*a*) G. Ribaud and E. Brun, Transmission de la Chaleur: Vol. I, Le Rayonnement Thermique, p. 81. J. et R. Sennac, Paris, 1948. 168 pp.
(*b*) A. Umur, G. V. Parmelee, and L. F. Schutrum, "Measurement of Angular Emissivity," *Heating, Piping, Air Conditioning*, 26, 135–40 (1954).

lated hemispherical emissive power of a semi-infinite transparent radiator, expressed in terms of an as yet unknown volume emissive power, and the hemispherical emissive power of a comparable opaque radiator, known in terms of the familiar Planck or Stefan-Boltzmann equations. The comparison leads to the result

$$j_\lambda = \gamma_\lambda n^2 W_{B\lambda}/\pi \qquad (6)$$

In this j_λ is the sought spectral volume emissive power, γ_λ the spectral absorption coefficient, and n the refractive index of the transparent material. $W_{B\lambda}$ is the hemispherical spectral emissive power of an ideal (or black-body) radiator, which can be computed by Planck's law (see equation (17), Appendix II).

This result is similar to McMahon's, except for the term n^2, which did not appear in his treatment.

(4) Note on the Emission of Radiation

A simple physical illustration of whence this term arises can be obtained by referring again to Fig. 2. This had been used to illustrate the fact that the radiation emitted was originally (i.e. inside the radiator) contained within a cone of semi-angle α_{crit}. It can be shown that the primary radiant flux within that cone is only the $(1/n^2)$th part of the total reaching the surface. The radiation emitted cannot, therefore, exceed the $(1/n^2)$th part of the total primary radiation reaching the surface from within. This result and the finding that the true volume emissive power is n^2 times that deduced by McMahon cancel one another insofar as the emitted radiation is concerned, which explains why McMahon's final results are correct, his results for volume emissive power notwithstanding.

Actually, some small part even of the radiation originally contained within the cone of semi-angle α_{crit} is also internally reflected, so that the radiant energy emitted, or the hemispherical flux leaving the radiator, must be slightly less than the $(1/n^2)$th part of that just within the surface. An external observer, aware only of the radiation emitted, cannot sense this higher flux just within the radiator. Nevertheless, the fact that this flux is greater than one might have expected can play an important role in the radiative redistribution of energy within the radiator, as was brought out by the amendment[5] of Kellett's earlier papers.

The foregoing considerations also show that if the radiator were permitted to emit not into air, which has a refractive index of 1.0, but into a medium of refractive index m (m less than n), the fraction of the primary radiant flux $W_{\lambda i\infty}$ emitted would change from a little less than $(1/n)^2$ to a little less than $(m/n)^2$. In other words, the rate of emission, as distinct from the rate of primary emission, increases approximately with the square of the refractive index of the medium into which emission is taking place.* It follows that the walls of a glass tank in contact with glass will emit more energy into the glass than one would expect from the Stefan-Boltzmann law, which applies for emission into air only.

This, essentially, is one of the premises from which Geffcken[6] started his treatment of radiative conduction in glass, which prompted the amendment of Kellett's earlier work.[3]

What we have done, thus far, is therefore little more than to have proved anew and discussed from an engineer's or glass technologist's point of view a fact well known to classical physicists, namely, that the rate of emission of radiation depends on the refractive index of the medium into which emission is taking place. This proof and discussion are presented in the hope that they may be of help to those who, like the writer, may have been somewhat baffled on learning of this fact, exemplified in Kellett's more recent note[5] by the substitution of σn^2 for the familiar constant σ of the Stefan-Boltzmann equation (see equation (19) of Appendix II). More specifically, the foregoing has served to establish an expression for volume emissive power which it is proposed to use in subsequent work.

III. The Emissivity of Transparent Sheets

With the volume emissive power known, we can turn to the emission of radiation by transparent sheets. As before, the primary radiation reaching either surface of a sheet is found first. The emitted radiation is then found by allowing for reflection and refraction effects at the two surfaces. The treatment differs from the preceding one principally in that multiple internal reflection must now be taken into account. Since present considerations are confined to isothermal sheets, results can conveniently be expressed in terms of emissivities.

(1) Primary Radiation in Transparent Sheets

The analysis illustrated by Fig. 1 is repeated, but equations (1) and (3) are now integrated between the limits $x = 0$ and $x = X$, X being the thickness of the sheet. The results obtained, after substitution for j_λ by equation (6), are

$$I_{\lambda\alpha X} = n^2 \frac{W_{B\lambda}}{\pi}\left(1 - e^{-\gamma_\lambda X/\cos\alpha}\right)\cos\alpha \qquad (7)$$

and

$$W_{\lambda iX} = n^2 W_{B\lambda}\left[2 - e^{-\gamma_\lambda X}(1-\gamma_\lambda X) - (\gamma_\lambda X)^2 Ei(-\gamma_\lambda X)\right] \qquad (8)$$

where the exponential-integral $Ei(-p)$ is defined by

$$Ei(-p) = -\int_p^\infty \frac{e^{-q}}{q}\, dq$$

It can be obtained from published tables.[9]

Equation (7) shows that, unlike primary radiation reaching the surface of a semi-infinite radiator, primary radiation reaching the surface of a transparent radiator of finite thickness does not obey Lambert's cosine law, the deviation being due to the term $e^{-\gamma_\lambda X/\cos\alpha}$. Equation (8) will not be used further. It is brought in only to introduce the exponential-integral function, which is to problems involving diffuse radiation what the simple exponential function e^{-x} is to problems involving unidirectional radiation. Thanks to the availability of tables, the treatment of diffuse radiation within one medium presents no difficulties. Only when one wishes to consider the reflection and refraction of diffuse radiation does it become necessary to change from algebra to more tedious graphical or numerical computations.

(2) Radiation Emitted by Transparent Sheets

Multiple internal reflection of the diffuse primary radiation reaching the surface of a plane-parallel transparent radiator is treated in Appendix III. It is shown that the radiation emitted from either side of the sheet is made up of the transmitted portion of the primary radiation directed toward that side and of the transmitted portion of radiation multiply reflected between the two surfaces.

Comparing the spectral intensities $I_{\lambda\beta X}$ and spectral flux $W_{\lambda eX}$ emitted by an isothermal transparent sheet of thickness X with the corresponding intensities and hemispherical emissive power of a black-body radiator at the same temperature, we define two emissivities. These are the *directional* spectral emissivity $\epsilon'_{\lambda X}$ and the *hemispherical* spectral emis-

* The greatest possible fraction of $W_{\lambda i\infty}$ that can be emitted is, of course, unity; and, for values of m greater than n, the emitted radiant flux is approximately equal to $W_{\lambda i\infty}$. This corresponds to emission into an optically denser medium, and changes of m/n (with $m > n$) affect only the angular distribution of the emitted radiation.

[9] (a) "Tables of Sine, Cosine, and Exponential Integrals," Federal Works Agency, W.P.A., New York, 1940.
(b) E. Jahnke and F. Emde, Tables of Functions. Dover Publications, New York, 1943.

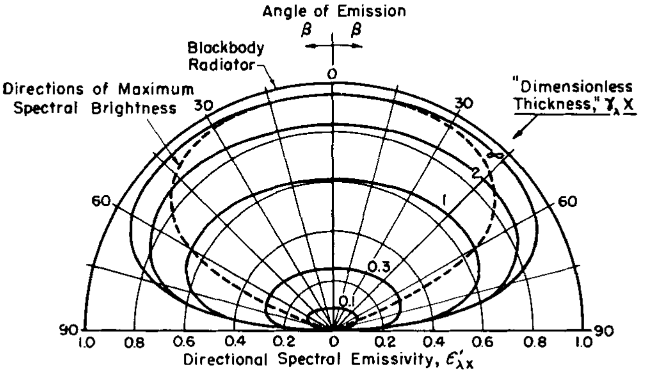

Fig. 4. Angular variation of the spectral emissivities of transparent sheets (n = 1.5).

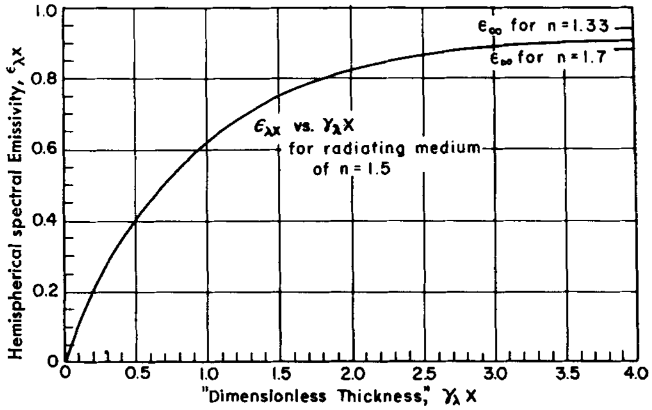

Fig. 5. Hemispherical spectral emissivity of transparent sheets.

sivity $\epsilon_{\lambda X}$, given by equations (23) and (24) of Appendix III.

Note that these results are in terms of a dimensionless product $\gamma_\lambda X$. This is the product of the actual thickness X of the sheet and of the absorption coefficient γ_λ of the material, which is generally wave-length dependent. Since for any given material at any given wave length one value of γ_λ applies, $\gamma_\lambda X$ might be termed a "dimensionless thickness."

Figure 4 shows the angular variation of spectral emissivities for sheets having various dimensionless thicknesses. This *directional* spectral emissivity $\epsilon'_{\lambda X}$ is the ratio that the intensity of radiation in any direction bears to the intensity of black-body radiation in the same direction. Since black-body radiation obeys Lambert's cosine law, i.e. is uniformly bright in all directions, Fig. 4 can also be regarded as a plot of the angular variation of the relative spectral "brightness" of radiating sheets. The outer curve refers to a radiating block of glass (n = 1.5) having an infinite dimensionless thickness. The normal emissivity of such a radiator is 0.96, and the emissivity is practically constant for angles of emission up to about 55°, beyond which it decreases. As has already been noted in reference to Fig. 3, the curve is very similar to the corresponding curves for most opaque nonmetallic materials. The other curves refer to successively thinner sheets of glass, the brightness of which decreases with decreasing thickness. The angular distribution of brightness also changes; as might be expected, the thinner sheets radiate relatively more strongly in oblique directions, as shown by the fact that the curves are progressively flatter. In contrast with thicker sheets and most opaque nonmetallic materials, these thinner sheets have mean, or hemispherical, emissivities greater than the corresponding normal emissivities.

Figure 5 shows *hemispherical* spectral emissivity as a function of dimensionless thickness. For a material having a given refractive index, the hemispherical spectral emissivity depends only on the magnitude of the product $\gamma_\lambda X$ and is practically constant for values of this greater than about 3.5. This gives a more definite meaning to the term "infinite thickness." Thus one can regard as "infinite" any transparent sheet for which $\gamma_\lambda X$ is greater than 3.5, regardless of whether this is so because of the great thickness of the sheet or the high absorption coefficient of its material for the radiation in question.

Since the refractive index of most glasses does not change significantly over the wave-length region of interest, the hemispherical emissivity of massive bodies of glass is seen to be independent of wave length. It is 0.91 for glasses having a refractive index of 1.5.

(3) Total Hemispherical Emissivity of Isothermal Sheets of Glass

In practical calculations of heat transfer one must know the total hemispherical emissivity, i.e. the emissivity taking into account radiation in all directions and of all wave lengths Calculation of this involves forming a mean of the hemispherical spectral emissivities, weighted according to the corresponding emissive powers of black-body radiation.

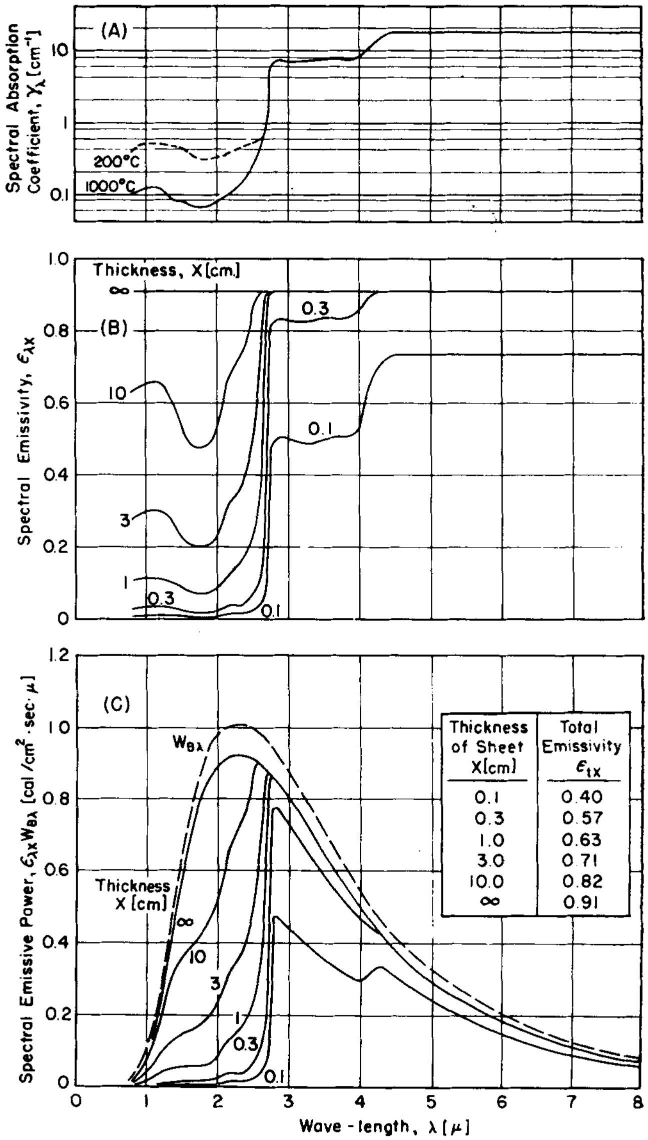

Fig. 6. Emissivity of sheets of window glass at 1000°C.

Figure 6 illustrates the steps in a graphical computation of the total hemispherical emissivities of sheets of window glass

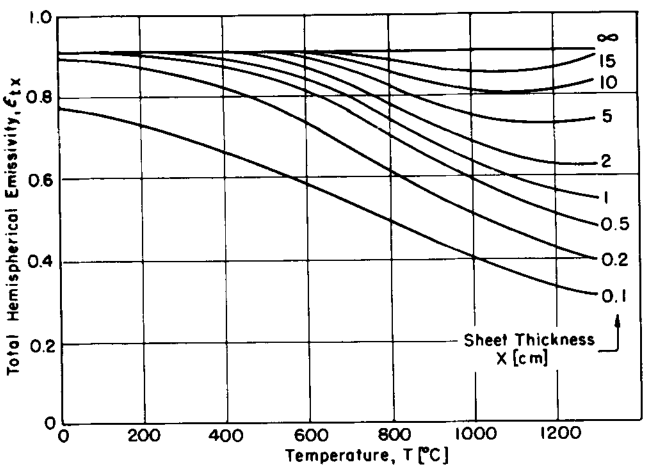

Fig. 7. Total hemispherical emissivity of window glass as a function of sheet thickness and temperature.

of various thicknesses and at a temperature of 1000°C. Figure 6 (*A*) shows the spectral variation of absorption coefficients, obtained from Neuroth.[10(a)] Figure 6 (*B*) shows the corresponding variation of the hemispherical spectral emissivities of sheets having various thicknesses, these emissivities being obtained from Fig. 5. Figure 6 (*C*) shows spectral emissive powers of black-body radiation at 1000°C., multiplied by the corresponding spectral emissivities of the various sheets. Areas under the curves of Fig. 6 (*C*) are a measure of the energy radiated at 1000°C. by sheets of window glass having various thicknesses.

A series of calculations following the foregoing procedure has led to values of the total hemispherical emissivity of sheets of window glass ranging in thickness from 1 mm. up and at temperatures between 20° and 1300°C. The results are exhibited in Fig. 7. While this applies to window glass only, similar curves can readily be obtained for any glass the spectral absorption coefficients of which are known. Data on absorption coefficients of various glasses at elevated temperatures have appeared in some recent German papers.[10, 11]

Window glass is typical of many other glasses in that its spectral absorption coefficients increase from relatively low values at wave lengths below about 2.7 μ to much higher values at longer wave lengths. Its absorption coefficients also vary with temperature, but the "step" in the vicinity of 2.7 μ appears to remain quite marked (see Fig. 6 (*A*)). As temperature changes, the corresponding spectral distribution of black-body radiation also changes, and at high temperatures progressively greater fractions of the radiant energy are emitted at shorter wave lengths. It follows that the character of the radiation emitted by a sheet of glass at any temperature is largely determined by the position of the spectral energy distribution curve of a black-body radiator at that temperature relative to the step in the absorption coefficient vs. wave-length curve. As far as the thickness of the sheet is concerned, it is evident from Fig. 5 that this affects (spectral) emissivities only if $\gamma_\lambda X$ is less than about 3.5.

With these two criteria in mind, three ranges of thickness may be distinguished. (1) Sheets for which $\gamma_\lambda X$ is greater than 3.5 at all wave lengths, both below and above 2.7 μ, are effectively "infinitely thick." Their total hemispherical emissivity depends on their refractive index only and is 0.91 for glasses having a refractive index of 1.5. (2) For very thin sheets the product $\gamma_\lambda X$ is less than 3.5 even for the high values of γ_λ obtaining at wave lengths longer than 2.7 μ. The emissivity of sheets in this range of thicknesses is very sensitive to changes in thickness. (Some uncertainty is attached to the computed emissivities of these very thin sheets because of their dependence on higher values of γ_λ, of which no measurements are available and which therefore had to be assumed to make these calculations possible.) (3) For sheets having intermediate thicknesses $\gamma_\lambda X$ is greater than 3.5 (and therefore $\epsilon_{\lambda X}$ is constant at 0.91) for radiation of relatively long wave lengths, and less than 3.5 in the wave-length region below 2.7 μ, so that the spectral emissivities at these lower wave lengths are still markedly thickness dependent. The net result is that the total hemispherical emissivity ϵ_{tX} is also thickness dependent, although to a lesser extent than for thinner sheets.

To illustrate some of these generalizations quantitatively, consider the emissivities of window glass at 200° and at 1000°C., as shown in Fig. 7. At 200°C. more than 99% of the energy of black-body radiation is emitted at wave lengths longer than 2.7 μ, to which window glass is relatively opaque. Consequently the total hemispherical emissivity of window glass at this temperature is quite high even for very thin sheets. It is 0.73 for a sheet only 0.1 cm. thick, and the limiting value of 0.91 obtains for a sheet only about 0.5 cm. thick. At 1000°C. more than 35% of the energy of black-body radiation is emitted in the wave-length region below 2.7 μ. Thus, while for wave lengths beyond 2.7 μ a sheet 0.5 cm. thick again has a *spectral* hemispherical emissivity of 0.91 (cf. Fig. 6 (*B*)), its *total* hemispherical emissivity is only 0.59. In fact, the total hemispherical emissivity remains discernibly less than 0.91 for sheets up to about 20 cm. thick. It may also be noted from Fig. 6 (*C*) that, although the peak intensity of emission by a thick body of glass always occurs at the same wave length as the peak intensity of black-body radiation, the same does not hold for the peak (or peaks) in the emission spectrum of thinner sheets.

IV. Radiation Within Transparent Sheets

We have now discussed that part of the primary and multiple internally reflected radiation that has been "emitted" across the surfaces of a transparent radiating sheet. In conclusion we might touch upon radiant fluxes within the sheet and thereby draw together the present considerations and treatments of radiative conduction in transparent materials.[3, 5, 6] These treatments have been restricted to the inner regions of massive bodies, in which the effects of the surface are not felt. They are not, therefore, applicable to thin sheets in which surface effects, such as emission, irradiation, and multiple internal reflection, can play a dominant part. No simple expression for radiative conduction can be obtained for these. However, by following the radiation remaining in the sheet in the same manner as was done for the radiation escaping from the sheet after each internal reflection and traversal, a complete picture of internal heat transfer conditions can be obtained.

A few simple generalizations can readily be made. In the unsteady state, e.g. during the cooling of a transparent sheet, one can consider two aspects of the internal radiation. One is to ask from what levels within the transparent sheet the emitted radiation originates. Evidently, the more any layer

[10] (*a*) Norbert Neuroth, "Der Einfluss der Temperatur auf die spektrale Absorption von Gläsern im Ultraroten, I" (Effect of Temperature on Spectral Absorption of Glasses in the Infrared, I), *Glastech. Ber.*, **25** [8] 242–49 (1952); *Ceram. Abstr.*, **1953**, April, p. 57*e*.

(*b*) Norbert Neuroth, "Der Einfluss der Temperatur auf die spektrale Absorption von Gläsern im Ultraroten, II" (Effect of Temperature on Spectral Absorption of Glasses in the Infrared, II), *Glastech. Ber.*, **26** [3] 66–69 (1953); *Ceram. Abstr.*, **1954**, March, p. 47*f*.

[11] Ludwig Genzel, "Messung der Ultrarot-Absorption von Glas zwischen 20° und 1360°C." (Measurement of Infrared Absorption of Glass Between 20° and 1360°C.), *Glastech. Ber.*, **24** [3] 55–63 (1951); *Ceram. Abstr.*, **1953**, March, p. 40*g*.

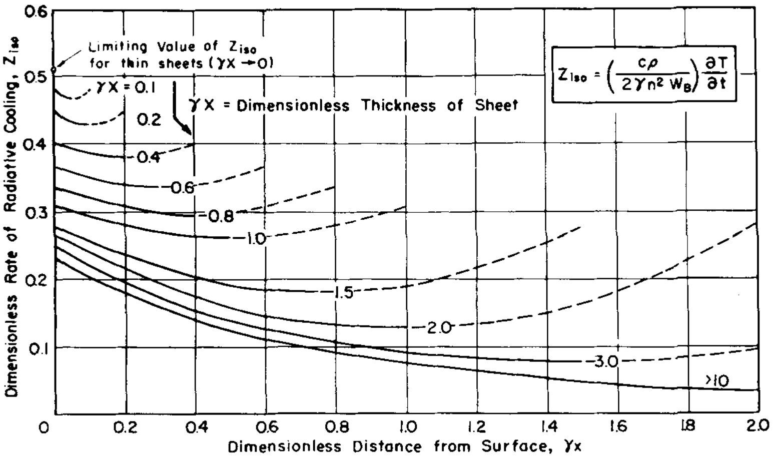

Fig. 8. Instantaneous rate of radiative cooling of an isothermal transparent sheet as a function of position and sheet thickness.

Briefly to illustrate this effect, Fig. 8 is presented without going into its proof or details. It shows the dimensionless rates of radiative cooling (due to the emission of monochromatic radiation) at various levels within initially isothermal transparent sheets, which are suddenly exposed to a cold environment. As is to be expected, thinner sheets cool faster than thicker sheets. The interesting point to note is that initially isothermal transparent sheets begin to cool simultaneously throughout their thickness. This is in contrast with initially isothermal opaque sheets, which, in the first instant of cooling, can lose heat from their surfaces only, not from their interior regions. This means that the more diathermanous a glass is, the smaller will be the temperature gradients created in a sheet of it by a given heat-treating operation. Figure 8 is conveniently simple to illustrate a point. However, the isothermal conditions for which it applies are somewhat restrictive. To give such a plot greater practical interest, more work must be done on the computation of local cooling rates within transparent sheets in which temperature gradients exist. This work is in progress, and it is hoped that it may shed light on the very interesting heat-transfer phenomena that are involved in such practical operations as annealing and tempering.

contributes to the emitted radiation, i.e. the nearer it is to the surface, the faster that layer will tend to cool. Secondly, partial reabsorption of the internal radiant flux, especially of that due to multiple internal reflections, will tend to equalize temperatures within the sheet. Under certain circumstances this process can be far more effective than ordinary thermal conduction. Thus, even where the external heat transfer between a sheet and its surroundings occurs principally by some mechanism other than radiation, the internal radiative redistribution of energy may have to be taken into account. In this connection it must be remembered that just within the transparent radiator the primary radiant flux is roughly n^2 times that emitted, and the total flux is yet higher—thanks to multiple internal reflection.

Acknowledgment

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APPENDICES

Appendix I. Intensity Relations for Reflection and Refraction

In Fig. 9 a narrow beam from the interior of a transparent radiator is shown to be partly reflected and partly refracted at the surface, where it passes through an area δS. The internal angle of incidence of the beam is α, its spectral intensity is $I_{\lambda\alpha}$, and it is contained within the small solid angle $\delta\Omega_i$.

$$\delta\Omega_i = \sin\alpha\ \delta\alpha\ \delta\varphi$$

$\delta\varphi$ being the "angular width" of the beam, measured perpendicularly to its plane of incidence. After refraction the beam emerges into a vacuum (or into air) at an angle of refraction β. Its spectral intensity is then $I_{\lambda\beta}$, and it is contained within the small solid angle $\delta\Omega_e$,

$$\delta\Omega_e = \sin\beta\ \delta\beta\ \delta\varphi$$

If τ' is the transmissivity of the surface for this beam, i.e., the fraction of the energy of the incident beam that is refracted across the surface, then, by an energy balance,

$$I_{\lambda\alpha}\,\delta\Omega_i\,\tau' = I_{\lambda\beta}\,\delta\Omega_e$$

so that

$$I_{\lambda\beta} = I_{\lambda\alpha}\,\tau'\,\frac{\delta\Omega_i}{\delta\Omega_e}$$

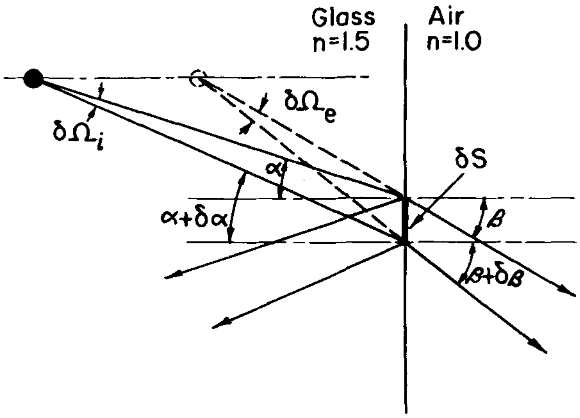

Fig. 9. Reflection and refraction of a divergent beam.

The angles of incidence and refraction, α and β, are related by Snell's law

$$\sin\beta = n\sin\alpha$$

Hence

$$\frac{\delta\Omega_i}{\delta\Omega_e} = \frac{1}{n^2}\frac{\cos\beta}{\cos\alpha}$$

The refracted intensity therefore becomes

$$I_{\lambda\beta} = I_{\lambda\alpha} \frac{\tau'}{n^2} \frac{\cos\beta}{\cos\alpha} \tag{9}$$

If the incident beam is one that originated in the interior of a semi-infinite transparent radiator, its intensity $I_{\lambda\alpha_\infty}$ will be given by equation (2), and the intensity of the corresponding refracted beam will be

$$I_{\lambda\beta_\infty} = \frac{j_\lambda}{\gamma_\lambda} \frac{\tau'}{n^2} \cos\beta \tag{10}$$

This is equation (5) of the main part of the paper.

The variation of τ' with α (or β) is considered next.

Fresnel's equations relate the amplitudes of the electric waves of the incident, reflected, and refracted beams. The energy associated with these beams is proportional to the squares of their respective amplitudes and also depends on the medium. For simplicity, one starts by comparing the energy of the incident and reflected beams, which are the two beams within the same medium. For these, Fresnel's equations lead to the following expressions for the *internal* reflectivity of the surface of a nonmetallic solid:

$$\rho'_\perp = \frac{\sin^2(\beta - \alpha)}{\sin^2(\beta + \alpha)} \tag{11a}$$

for radiation polarized perpendicularly to plane of incidence,

and
$$\rho'_\parallel = \frac{\tan^2(\beta - \alpha)}{\tan^2(\beta + \alpha)} \tag{11b}$$

for radiation polarized parallel to plane of incidence.

As the radiation emitted within the glass is not polarized, the effective reflectivity ρ' is given by

$$\rho' = \frac{1}{2}(\rho'_\perp + \rho'_\parallel) \tag{12}$$

Finally, as no absorption occurs in the infinitesimally thin surface region in which reflection takes place, the corresponding transmissivity of the surface is

$$\tau' = 1 - \rho' \tag{13}$$

The transmissivity τ' is thus seen to depend on the refractive index of the material and on the angle of incidence of the radiation (see Fig. 3). τ' is independent of wave length for any material the refractive index of which may be regarded as constant, i.e., for a material of zero dispersion. It should also be noted that the foregoing expression for τ' applies only for initially unpolarized radiation.

Appendix II. Determination of Volume Emissive Power

Insofar as $W_{\lambda i_\infty}$ represents the spectral hemispherical flux reaching the surface from within a semi-infinite transparent radiator, let $W_{\lambda e_\infty}$ represent the spectral hemispherical flux refracted across the surface, i.e. the flux "emitted." It is given by

$$W_{\lambda e_\infty} = 2\pi \int_0^{\pi/2} I_{\lambda\beta_\infty} \sin\beta \, d\beta = \frac{\pi j_\lambda}{\gamma_\lambda} \frac{\tau^*}{n^2} \tag{14}$$

where

$$\tau^* = 2\int_0^{\pi/2} \tau' \sin\beta \cos\beta \, d\beta \tag{15}$$

Similarity of the angular distributions of radiation emitted, respectively, by an opaque radiator and by a semi-infinite transparent radiator has suggested the hypothesis that these two radiators may be identical. Hence the emissive power $W_{\lambda e_\infty}$ should be equal to that of an opaque radiator, $W_{R\lambda}$. The latter is given by

$$W_{R\lambda} = \epsilon_{R\lambda} W_{B\lambda}$$

where $W_{B\lambda}$ is the emissive power of an ideal (or black-body) radiator and $\epsilon_{R\lambda}$ is the hemispherical spectral emissivity of the real radiator.

By Kirchhoff's law, the hemispherical spectral emissivity of an opaque body is given by

$$\epsilon_{R\lambda} = 1 - \bar{\rho}$$

where $\bar{\rho}$ is the hemispherical reflectivity of its surface, the bar indicating that this applies to (diffuse) radiation reaching the surface from the optically less dense medium. If this diffuse radiation obeys Lambert's cosine law, one obtains

$$\bar{\rho} = 2\int_0^{\pi/2} \bar{\rho}' \sin\bar{\alpha} \cos\bar{\alpha} \, d\bar{\alpha}$$

where $\bar{\alpha}$ is now the angle of incidence in vacuum (or air) and $\bar{\rho}'$ is the angle-of-incidence-dependent reflectivity. For a nonmetallic and nonscattering material, $\bar{\rho}'$ is given by equations (11) and (12), with $\bar{\alpha}$ taking the place of β and $\bar{\beta}$ taking the place of α. Comparing this with equations (15) and (13), it is seen that

$$\bar{\rho} = 1 - \tau^*$$

It follows that

$$\epsilon_{R\lambda} = \tau^*$$

Hence, if

$$W_{\lambda e_\infty} = W_{R\lambda}$$

then

$$j_\lambda = \gamma_\lambda n^2 W_{B\lambda}/\pi \quad [\text{cal./cm.}^3\cdot\text{sterad.}\cdot\text{sec.}\cdot\mu] \tag{16}$$

This is the desired expression for spectral volume emissive power, equation (6) of the main part of the paper.

The spectral emissive power of a black-body radiator, $W_{B\lambda}$, can be calculated by Planck's law

$$W_{B\lambda} = \frac{c_1}{\lambda^5(e^{c_2/\lambda T} - 1)} \quad [\text{cal./cm.}^2\cdot\text{sec.}\cdot\mu] \tag{17}$$

in which T = temperature of radiator [°K.].
$c_1 = 8.94 \times 10^3$ cal.$\cdot\mu^4$/cm.$^2\cdot$sec.
$c_2 = 14.39 \times 10^3$ $\mu\cdot$°K.

Unlike the index of refraction, the absorption coefficient γ_λ varies markedly with wave length, so that one cannot readily speak of a "gray" absorber. If one could, one could also speak of a *total* volume emissive power j, which would be related to the total emissive power of a black-body radiator in the same manner as the *spectral* volume emissive power j_λ is related to the spectral emissive power of a black body. Thus, j would be given by

$$j = \gamma n^2 W_B/\pi \quad [\text{cal./cm.}^3\cdot\text{sterad.}\cdot\text{sec.}] \tag{18}$$

where W_B is the total emissive power of a black-body radiator, given by the Stefan-Boltzmann law

$$W_B = \sigma T^4 \quad [\text{cal./cm.}^2\cdot\text{sec.}] \tag{19}$$

in which σ is the Stefan-Boltzmann constant, 1.355×10^{-12} cal./cm.$^2\cdot$sec.$\cdot$°K^4.

Appendix III. Multiple Internal Reflection of Radiation Emitted Within a Plane-Parallel Sheet

Initially unpolarized radiation becomes partly polarized upon reflection, the degree of polarization depending on the angle of incidence. Upon multiple reflection between parallel surfaces, the degree of polarization becomes more marked. Thus, although an initially unpolarized beam within a plane-parallel sheet will have the same angle of incidence at each of many internal reflections, its reflectance ρ' and transmittance τ' will vary with its changing state of polarization. However, $\rho'_\perp$, $\rho'_\parallel$, $\tau'_\perp$, and $\tau'_{\|}$ are constants for any given angle of incidence α. It will be convenient, therefore, to consider separately beams of radiation polarized, respectively, perpendicularly and parallel to the plane of incidence.

Consider a monochromatic beam of perpendicularly polarized primary radiation that reaches one face of a plane-parallel sheet at an angle of incidence α and with an intensity $I_{\lambda\alpha\perp}$. A fraction $\tau'_{\perp}$ of its energy is transmitted across the surface, and a fraction $\rho'_{\perp}$ is reflected. The transmitted fraction emerges at an angle of "emission" β, and its intensity is given by equation (9) as

$$I^{\circ}_{\lambda\beta\perp} = \frac{\tau'_{\perp}}{n^2}\frac{\cos\beta}{\cos\alpha} I_{\lambda\alpha\perp}$$

The internally reflected beam, having an initial intensity $I_{\lambda\alpha\perp}\rho'_{\perp}$, is attenuated as it travels a distance $X \sec \alpha$ before being again partially reflected at the opposite face. It is further attenuated as it crosses the sheet again to reach the first surface for a second time. Its intensity is then

$$I_{\lambda\alpha\perp}(\rho'_{\perp})^2 e^{-2\gamma_\lambda X \sec\alpha}$$

and of this a fraction $\tau'_{\perp}$ is again transmitted and $\rho'_{\perp}$ reflected. The intensity of the transmitted part of the twice-reflected beam is (again by equation (9))

$$I''_{\lambda\beta\perp} = \frac{\tau'_{\perp}}{n^2}\frac{\cos\beta}{\cos\alpha} I_{\lambda\alpha\perp}(\rho'_{\perp})^2 e^{-2\gamma_\lambda X \sec\alpha}$$

The internally reflected part continues to traverse the sheet until it has become completely attenuated, partly by absorption and partly by transmission across the two surfaces.

Thus, of the perpendicularly polarized primary radiation initially directed toward one face, an amount of intensity

$$\frac{\tau'_{\perp}}{n^2}\frac{\cos\beta}{\cos\alpha} I_{\lambda\alpha\perp}[1+(\rho'_{\perp})^2 e^{-2\gamma_\lambda X \sec\alpha} + (\rho'_{\perp})^4 e^{-4\gamma_\lambda X \sec\alpha} + \ldots]$$

emerges across that face. At the same time, of the primary radiation initially directed toward the opposite face, an amount of intensity

$$\frac{\tau'_{\perp}}{n^2}\frac{\cos\beta}{\cos\alpha} I_{\lambda\alpha\perp}[(\rho'_{\perp}) e^{-\gamma_\lambda X \sec\alpha} + (\rho'_{\perp})^3 e^{-3\gamma_\lambda X \sec\alpha} + \ldots]$$

also emerges across the first face. Thus, the total emergent intensity from either face is

$$\begin{aligned} I_{\lambda\beta\perp} &= \frac{\tau'_{\perp}}{n^2}\frac{\cos\beta}{\cos\alpha} \times \\ &\quad I_{\lambda\alpha\perp}[1 + (\rho'_{\perp}) e^{-\gamma_\lambda X \sec\alpha} + (\rho'_{\perp})^2 e^{-2\gamma_\lambda X \sec\alpha} + \ldots] \\ &= \frac{1}{n^2}\frac{\cos\beta}{\cos\alpha} I_{\lambda\alpha\perp} \frac{\tau'_{\perp}}{1 - (\rho'_{\perp}) e^{-\gamma_\lambda X \sec\alpha}} \end{aligned}$$

A similar expression is obtained for the primary radiation polarized parallel to the plane of incidence. Since the primary radiation in a sheet of an isotropic material, such as glass, is not polarized, the intensities $I_{\lambda\alpha\perp}$ and $I_{\lambda\alpha\|}$ are equal to one another and to $^1/_2 I_{\lambda\alpha X}$, which is given (for isothermal sheets) by equation (7).

Hence the intensity of radiation emitted by an isothermal sheet of thickness X is given by

$$I_{\lambda\beta X} = \frac{W_{B\lambda}}{\pi}(1 - e^{-\gamma_\lambda X \sec\alpha})\Upsilon' \cos\beta \qquad (20)$$

where

$$\Upsilon' = \frac{1}{2}\left[\frac{\tau'_{\perp}}{1 - (\rho'_{\perp}) e^{-\gamma_\lambda X \sec\alpha}} + \frac{\tau'_{\|}}{1 - (\rho'_{\|}) e^{-\gamma_\lambda X \sec\alpha}}\right] \qquad (21)$$

From this, integrating over the hemisphere, the emergent radiant flux is obtained as

$$\begin{aligned} W_{\lambda e X} &= 2\pi \int_0^{\pi/2} I_{\lambda\beta X} \sin\beta \, d\beta \\ &= W_{B\lambda} \int_0^{\pi/2} (1 - e^{-\gamma_\lambda X \sec\alpha})\, \Upsilon' \sin\beta\cos\beta \, d\beta \qquad (22) \end{aligned}$$

The corresponding directional and hemispherical spectral emissivities of an isothermal sheet of thickness X are given by

$$\epsilon'_{\lambda X} = \frac{I_{\lambda\beta X}}{I_{B\lambda\beta}} = \frac{\pi I_{\lambda\beta X}}{W_{B\lambda}\cos\beta} = (1 - e^{-\gamma_\lambda X \sec\alpha})\Upsilon' \qquad (23)$$

and

$$\epsilon_{\lambda X} = \frac{W_{\lambda e X}}{W_{B\lambda}} = \int_0^{\pi/2} (1 - e^{-\gamma_\lambda X \sec\alpha})\, \Upsilon' \sin\beta\cos\beta\, d\beta \qquad (24)$$

This integral is best evaluated graphically.

List of Symbols and Typical Units

c	Specific heat	cal./gm.°C.
e	2.718 (base of natural logarithms)	
I	Intensity of radiation	cal./cm.²·sterad.·sec.
I_λ	Spectral intensity of radiation	cal./cm.²·sterad.·sec.·μ
j_λ	Spectral volume emissive power	cal./cm.³·sterad.·sec.·μ
L	Length of path of radiation	cm.
m	Refractive index of medium into which emission takes place	
n	Refractive index in general, of radiator in particular	
Q_λ	Spectral rate of flow of radiant energy	cal./sec.·μ
r	Cylindrical coordinate	cm.
S	Surface area	cm.²
T	Temperature	°K.
V	Volume	cm.³
W	Radiant flux, or hemispherical emissive power	cal./cm.²·sec.
W_λ	Spectral radiant flux, or hemispherical spectral emissive power	cal./cm.²·sec.·μ
X	Thickness of transparent sheet	cm.
x	Position coordinate	cm.
α	Angle of incidence (within radiator)	
β	Angle of refraction or emission	
γ_λ	Spectral absorption coefficient	cm.⁻¹
ϵ	Hemispherical emissivity; ϵ', directional emissivity	
λ	Wave length	μ
ρ	Mean reflectivity of a surface for diffuse radiation; ρ', directional reflectivity	
ρ	(In Fig. 8) density	gm./cm.³
σ	Stefan-Boltzmann constant, 1.355×10^{-12}	cal./cm.²·sec.·°K.⁴
τ	Mean transmissivity of a surface for diffuse radiation; τ', directional transmissivity	
Υ'	Effective directional transmissivity (see equation (21))	
φ	Angle in cylindrical coordinate system	
ω	Solid angle subtended by the receiver of radiation at the source	steradian
Ω	Solid angle subtended by the source of radiation at the receiver	steradian

Subscripts

B	Of a black-body (or ideal) radiator
e	Refers to flux emitted
i	Refers to flux incident on inside of surface of radiator
R	Of a real radiator
t	Total, referring to radiation of all wave lengths
X	Of a sheet of thickness X
α	At angle of incidence α, or associated with incident beam (within radiator)
β	At angle of refraction (emission) β, or associated with emitted beam
λ	Spectral, i.e. pertaining to radiation of wave length λ; and, in the case of energetic quantities, referred to unit interval of wave length (see units of I and I_λ)
∞	Of a semi-infinite radiator
$\perp$	For radiation polarized perpendicularly to plane of incidence
$\|$	For radiation polarized parallel to plane of incidence