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The Velocities of Light*

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The definitions, the physical significances, the interrelationships, and the observability of seven velocities of light are discussed. One of the seven, the centrovelocity, is a new velocity which is introduced here. It is suggested that this velocity can be used for the description of the transport of electromagnetic radiation since it does not have any of the short comings of the standard definitions of the group velocity or the velocity of energy transport.

A. INTRODUCTION

The velocity of light is one of the most fundamental of all physical constants. At the present time, while there are a number of excellent reviews[1–4] of the experimental determination of the velocity of light, there is no study of the velocity of light as a concept in the literature. While there is very little difference in the magnitude of the velocities of light in weakly dispersive media there exist a large class of modern problems where a significant amount of dispersion is present and there is a noticeable difference in the velocities. These cases are due mainly to the development of the laser and the multitude of new types of propagation situations that have arisen, such as the propagation of pulses in an amplifying medium, in a nonlinear medium, and others.

We will attempt to clarify the basic characteristics or nature of the concept of the velocity of light. We will find, as is well known, there is more than one velocity of light. We will attempt to ascertain which of these are quantities which are experimentally observable, what are their charactersitics, and how are they interrelated. Thereby, we hope to gain a deeper understanding of this topic. In particular, we introduce a new definition for a velocity of light which is a decided improvement over some in current usage.

We will start with a detailed discussion of the definitions and nature of the seven velocities of light.

B. THE VELOCITIES OF LIGHT

Normally when one makes reference to "the velocity of light" one is referring to the phase velocity of plane waves of light in a vacuum. However, there are times when one speaks of the velocity of light, but does not refer to this. There seems to be at least six velocities of light in conventional use. To these six we will add one new definition of our own, i.e., the centrovelocity. In this section we shall discuss the most precise definitions we have been able to obtain or formulate for these velocities:

1. the phase velocity,
2. the velocity of energy transport,
3. the group velocity,
4. the relativistic velocity constant,
5. the ratio of units velocity,
6. the signal velocity,
7. the centrovelocity.

1. Phase Velocity

The first velocity associated with light or electromagnetic fields which we shall discuss is the well known phase velocity v_p. Let us assume we have a monochromatic wave of the form

$$\psi(r, t) = A(r) \cos[\omega t - g(r)], \tag{1}$$

where $g(r)$ is a real scalar function. The phase velocity is defined as[5]

$$v_p(r) = \omega / |\nabla g(r)|. \tag{2}$$

Its direction may be taken the same as $\nabla g(r)$.

For a monchromatic plane wave the phase velocity is given by

$$v_p = \omega / k. \tag{3}$$

Equation (3) is probably the most widely used definition for phase velocity. However, Eq. (2) is preferred because it holds for any monochromatic field while Eq. (3) holds only for plane waves.

If one chooses a particular value of the phase for a monochromatic wave at some position and time, the phase velocity is the velocity with which this constant value of the phase travels through space. Thus, the phase velocity determines the phase of a monochromatic wave in space and time if the phase is known at some position at a given time. However, one should not assume that since the phase of a monochromatic component of a wave packet has had time to reach a given space–time point, the field or any observable quantity has arrived at that point. This is because there is no observable physical quantity associated with the phase of a light wave. Furthermore, we may have the phase determined at a certain space point by the phase velocity, but the amplitude of the wave may be zero there. Thus the phase is actually not defined at the point. Thus even though the phase velocity implies a phase at a given point there need not be a nonzero field at that point. We therefore see that the phase velocity is not the velocity of an observable physical object. For plane monochromatic waves it gives the relationship between ω and k.

We wish to emphasize that the phase velocity of light can not be directly measured by a time of flight method. It must always be the result of a calculation other than distance traveled divided by the time of flight. By this we mean one cannot observe a fixed value of the phase at point A and time its propagation to point B. In the optical range and above, there has been no direct, accurate measurement of the phase velocity.

A phase velocity cannot be attributed to a wave packet or to any wave except a monochromatic wave. This is due to the fact that any wave form consisting of more than one monochromatic wave does not have a unique frequency. However, some physical realizable wave packets can be treated as monochromatic waves to a very good approximation.

2. Velocity of Energy Transport

We now wish to consider the velocity of energy transport or, as it is often called, the ray velocity. In a loss-free region the velocity of energy transport is defined as[5–8]

$$\mathbf{v}_e = \mathbf{S}/W, \tag{4}$$

where $\mathbf{S}$ is the Poynting vector and W is the energy density. There are some practical difficulties with this definition. The relationship between the field vectors and the energy flow or the energy density is subject to a degree of arbitrariness.[5] If one assumes that the energy flow is given by Poynting vector, it can be determined experimentally. However, the energy density cannot be measured experimentally. Therefore, there cannot be a direct determination of $\mathbf{v}_e$ by the experimental observation of $\mathbf{S}$ and W. However, the most serious difficulty, from an experimentalist point of view, with the above definition is that it does not define when the energy has arrived. Therefore, one cannot measure by a time of flight method or any other method the velocity of energy transport as defined by Eq. (4).

We conclude that the definition of ray velocity as given by Eq. (4) does not correspond to the propagation of a real observable physical quantity. However, this should not be taken to mean that the concept of velocity of energy transport is not useful.

3. Group Velocity

(a) *Standard Definition*

Almost any wave packet may be written as a Fourier integral such as

$$\psi(\mathbf{r}, t) = \int_0^\infty A_\omega(\mathbf{r}) \cos[\omega t - g_\omega(\mathbf{r})] d\omega. \tag{5}$$

We have for the definition of the group velocity for such a wave packet that[5]

$$v_g(\mathbf{r}) = |\nabla(\delta g_\omega(\mathbf{r})/\delta\omega|_{\bar{\omega}})|^{-1}, \tag{6}$$

where $\bar{\omega}$ is the mean frequency of the signal. We note that $\bar{\omega}$ is not uniquely defined. We may use a range of values for $\bar{\omega}$ without effecting the results of this section. We may treat the group velocity as a scalar or let it have the direction of $\nabla(\delta g_\omega/\delta\omega)$. For the case when $g_\omega(\mathbf{r}) = \mathbf{k}\cdot\mathbf{r}$ this reduces to

$$v_g = \delta\omega/\delta k|_{\bar{\omega}}. \tag{7}$$

By its nature, the group velocity is a mathematical entity which may not have any real physical significance associated with it. There is no physical particle, mass, energy, or signal which necessarily travels at the group velocity. This is clearly the case in a region of anomalous dispersion as well as for a region of amplification. In a region of absorption v_g may become negative, zero, or infinite.[9] In fact Eq. (6) may no longer yield a unique value for v_g. To see this one need only consider the case where one wave packet enters, and after passing a distance in the medium, there are several packets separated in space at a given instance. In this case one cannot define *a* group velocity by the use of Eq. (6).

A wave form which has more than one maximum at a given time may have one, two, or more maxima at some later time. There is no law of conservation of the number of such maxima. Furthermore, we cannot associate with this maximum any unique physical entity which we can use as a tag and thereby follow its progress.

The reported demonstrations[7,8] of the equality of the group velocity and the velocity of energy transport have been limited to special cases, the most important restriction being that the medium is loss free. We note that this restiction is equivalent to the abandonment of the principle of causality.[10]

We wish to show that the standard definition of the group velocity fails to describe the motion of the peak of an arbitrary pulse in a region of anomalous dispersion. According to the standard definition the group velocity in a region of anomalous dispersion can exceed c, go to positive infinity, negative infinity, and assume a large range of negative values.[9] Needless to say, the behavior of the group velocity in this region is not consistent with what one would consider reasonable.

In the derivation of the expression for the group velocity found in modern texts,[5] the position of the maximum of the pulse is given by

$$t=\delta g_\omega(r)/\delta\omega\,|_{\bar{\omega}}. \tag{8}$$

If we have

$$g_\omega(r)=n\omega x/c, \tag{9}$$

it follows that

$$t=c^{-1}(\delta n\omega/\delta\omega)_{\bar{\omega}}x. \tag{10}$$

The standard definition of the group velocity fails whenever Eq. (10) yields a value for the time position of the maximum such that

$$t-x/c<0 \tag{11}$$

or

$$(\delta n\omega/\delta\omega)\,|_{\bar{\omega}}-1<0 \tag{12}$$

since we know the field is zero for all t's that satisfies Eq. (11).[10]

Even though the standard definition of the group velocity implies that the peak of the wave group has arrived, it has not. The standard group velocity fails because in the derivation an assumption was made that in general is not true in a region of anomalous dispersion. That is it was assumed that

$$k(\omega)-k(\bar{\omega})\approx(\omega-\bar{\omega})[\delta k(\omega)/\delta\omega]|_{\bar{\omega}}. \tag{13}$$

This approximation is not in general valid in a region where there is a resonance. This is why the standard expression for the group velocity does not describe the motion of the maximum of the pulse in a region where one has gain or absorption.

A general expression for the group velocity, i.e., the velocity of the maximum of the intensity of the pulse, in a region of anomalous dispersion is not readily apparent. The conventional one is clearly unacceptable. Furthermore, the group velocity of a pulse is a function of the gain or absorption, the depth in the medium, and the pulse shape.[11] Thus, the group velocity is a much more complex quantity than it is normally assumed to be.

(*b*) *A New Definition*

Because of the distortion due to dispersion, a new definition has recently been proposed for the group velocity.[12] It was proposed that the group velocity be the velocity of motion of the temporal center of gravity of the amplitude of the wave packet. Under this definition the group velocity could be written as

$$v_{gt}=\left|\nabla\left(\int_{-\infty}^{\infty}t\,|A(r,t)|\,dt\Big/\int_{-\infty}^{\infty}|A(r,t)|\,dt\right)\right|^{-1} \tag{14}$$

For a quasimonochromatic pulse, this definition reduces to that of Eq. (6). This definition has the advantage that for any case there exists a unique temporal center of gravity as long as the integrals converge. This is true even in the case where the original pulse splits into several parts. Furthermore, the pulse need not be quasimonochromatic as the previous definition required.

For the experimentalist, an additional amount of work may be required to determine the temporal center of gravity of the amplitude, but there is seemingly no serious difficulty.

4. The Relativistic Velocity Constant

For the four-dimensional space–time world of Minkowski we have the invariant quantity[13]

$$ds^2=dx^2+dy^2+dz^2-c_1^2dt^2. \tag{15}$$

The constant c_1 which has the dimensions of velocity is the quantity we call the relativistic velocity constant.[14,15] With this notation, the Lorentz transformation would be written in the usual form but with c replaced by c_1. Therefore, one would be led to the customary expression for the transformation of velocities, except again we must replace c by c_1. Therefore, if the velocity is c_1 in one frame, it is c_1 in all frames. Also, by use of the standard arguments[16] c_1 is the maximum velocity for a signal and the least upper limit for the velocity of particles with nonzero rest mass.

Historically, c_1 was taken as "the" velocity of light. However, one can conceive of a space–time geometry which is described by Eq. (15), but for which c_1 is not the velocity of propagation of a light signal, although it is very nearly equal to it.

5. The Ratio of Units Velocity

One may write two of Maxwell's equations in the following form

$$\nabla\times\mathbf{H}-c_r^{-1}\dot{\mathbf{D}}=(4\pi/c_r)\mathbf{J}, \qquad (16)$$

$$\nabla\times\mathbf{E}+\dot{\mathbf{B}}/c_r=0. \qquad (17)$$

We have used Gaussian units. The constant c_r is equal to the ratio of the unit of charge in the electrostatic units to that of the unit of charge in the electromagnetic system of units. This quantity is what we have called the ratio of unit velocity. This is not a velocity in the physical sense, but merely a universal physical constant which has the dimensions of velocity.

For free space where the solution of Maxwell's equations yield solutions which are monochromatic plane waves, the phase velocity for any frequency component is equal to the ratio of units velocity.

Historically, it was observed that Maxwell's equations for the vacuum were invariant under a Lorentzian transformation. For this demonstration the velocity term in Maxwell's equations and the velocity term in the Lorentzian transformation were taken as identical. Therefore, historically it was assumed that

$$c_r=c_1.$$

While there is seemingly no evidence which casts doubt upon this identity, it is of interest to see if these quantities can be independently determined. Both c_r and c_1 can in principle be determined experimentally; no direct experimental determination of c_1 has been made to date.

6. The Signal Velocity

To define a signal velocity one must first define a signal. If a change occurs in one body and it influences or produces a change in another body after the lapse of a certain interval of time, we say there exists an interaction between these bodies. Interactions traveling from one particle to another are often called "signals."[17] If the interaction is electromagnetic in nature then we would have a light signal. Because we wish to say the signal has a beginning we require that it be localized in time. Therefore, a signal is normally[9] defined as an isolated wave form of some arbitrary shape. Nothing should precede the signal which can be used to detect the coming of the signal. Thus the wave form must be zero before it starts. This definition corresponds in many respects to the concept of "cause" as used by Hilgevoord.[18]

Operationally, one may define the signal velocity in the following manner: The time at which the first nonzero part of the signal passes point A is noted. Likewise it is noted at some point B further along the path of propagation. The velocity is determined in the normal manner by dividing the distance from A to B by the elasped time.

To observe the first nonzero part of a signal one would need an infinitely sensitive detector. Since we have assumed that the field is zero for a finite interval, the signal must contain arbitrarily large frequencies. Therefore, our ideal detector would have to have the same infinite sensitivity for all frequencies. In practice we do not have detectors like this. A very widely used device is a photodetector. If the signal one wishes to observe is reasonably monochromatic, this device will give a good approximation of the pulse shape of the signal. However, if the sensitivity of the detector varies significantly over the spectral width of the incident signal, the detector will not give a good description of the incident signal.

Since an ideal detector is not available, it has become customary to choose some other criterion than the one given above to determine when a signal has arrived. Basically, these criteria fall into two types. The first has the form that the signal is taken to have arrived when it has reached $1/\alpha$ of its maximum amplitude where $\alpha\geq1$. The second type is that the signal has arrived when the stationary portion of the signal is equal in amplitude to the transient portion. Both of these criteria can lead to difficulties. When there is no dispersion present either of the above criteria yields consistent results. However, they both fail in a region of gain or absorption.

Let us consider the results of using the first criterion when dispersion alters the shape of the pulse significantly. Let us assume we have a symmetrical pulse entering a dispersive medium. The maximum of this pulse can be shifted forward or backward. Therefore, the signal velocity of these two cases will be different. The amount of difference will depend upon the value of α chosen. Clearly this is undesirable.

Possibly the most important objection to this criterion is that one must see the entire pulse before he can decide when it has arrived. Considering the fact that there are media which have gain it is possible to be able to detect the arrival of a pulse a significant time before this criterion says it has arrived.

If one uses a very large value of α the arrival of the signal may be indicated by the precursors. Thus the signal would be said to have arrived long before the main body of the pulse had arrived.

To avoid this difficulty the second criterion was introduced. The difficulty with the second criterion is that in some cases the stationary portion of the signal is always smaller than the transient portion.[19] Thus, if one can see the transient portion, and this criterion implies one can, the signal never arrives by this criterion even though we can see part of it, i.e., the transient part. Finally, if the incident frequency is in a region of anomalous dispersion, it is very difficult if not impossible to distinguish between the transient and stationary portions. Thus, this criterion fails.

The difficulty of arriving at a workable definition for the signal velocity is that a pulse of radiation is not a point, i.e., the motion of the pulse cannot be equated to the motion of a point associated with the pulse.

Let us reconsider the definition of a signal. Let us suppose that the wave form is zero for a finite interval before the signal is nonzero. To have the field zero for a finite interval the signal must contain arbitrarily large frequencies. The physical interpretation of these high frequency components is awkward. However, if we relax our demands that the wave form be precisely zero to that the resultant amplitude be smaller than some value δ, we need not have arbitrarily high frequencies. In a given case the value of δ will be set by the detection equipment used, the background noise, etc.

This leads us to the following criterion for determining the arrival of a signal: When one can detect an intensity greater than δ the signal is said to have arrived. It is believed that this criterion does not lead to the difficulties of the other criteria for macroscopic cases.

So far the discussion has been concerned only with macroscopic signals, that is, signals large enough so that we may sample them at two different points extracting a small portion of the total energy each time. Under this definition a macroscopic signal can consist of a single photon or many. The case where a single photon constituted a signal would be when the frequency was very great, such as for γ rays. In any case the energy extracted in the measuring process must be very small. For microscopic signals in the optical range there is seemingly no way of determining experimentally the signal velocity.

Finally, we would like to point out that whenever the light field satisfies the equation

$$\nabla^2\psi_\omega-[n^2(\omega)/c^2]\psi_\omega=0 \tag{18}$$

it can be shown[20] that the signal velocity is less than or equal to the maximum phase velocity of any frequency present in the signal. This result restricts the signal velocity to c or less in free space. However, in a region of gain or absorption this result does not restrict the signal velocity to values less than c.

7. The Centrovelocity

We would like to offer a new definition for a velocity of light which has some unique advantages. We offer the following definition

$$v_c=\left|\nabla\left(\int_{-\infty}^{\infty}tE^2(\mathbf{r},t)\,dt\Big/\int_{-\infty}^{\infty}E^2(\mathbf{r},t)\,dt\right)\right|^{-1}, \tag{19}$$

where $E(\mathbf{r},t)$ is the real amplitude of the microscopic electric field of the electromagnetic radiation field. The ratio of the two integrals yields the temporal center of gravity of the intensity. We see that this velocity describes the motion of the center of gravity of the intensity and thereby the motion of the center of gravity of the energy associated with the radiation field. Thus we have a velocity analogous to the center of mass velocity of dynamics. The centrovelocity describes the flow of the center of gravity of the pulse or pulses in a manner similar to how the motion of the center of mass is given by the center of mass velocity in classical dynamics.

In general, one may define for a given direction of propagation, a centrovelocity for the two states of polarization. Thus, one can define a centrovelocity for each of the two orthogonal planes of polarization or for the two senses of circular polarized light. In general, the different states of polarization can have different centrovelocities.

Equation (19) reduces to Eq. (6) for a quasimonochromatic pulse. However, unlike the group velocity, the centrovelocity is not restricted to a quasimonochromatic pulse or to a situation which one has always only a single pulse. Also, while it may or may not be always equal to the velocity of energy transport as given by Eq. (4), it can be experimentally determined in all cases. It is defined such that one can use a time of flight method to measure it. Also being the velocity of the center of gravity of the energy it tells at least part of the story of the flow of the energy. We would like to point out that the centrovelocity is well behaved in a region of anomalous dispersion while the group velocity is not.

Therefore, we see that the centrovelocity can be used to characterize the propagation of a pulse of radiation in all cases regardless of the dispersion present. It also yields an experimentally observable quantity which describes in part the flow of the energy associated with the pulse. The use of the centrovelocity in lieu of the group velocity and the velocity of energy transport would bring more order and simplification into this subject. Therefore, we believe the centrovelocity is a useful concept.

C. CONCLUSIONS

We have now completed our analysis of the velocities of light. We have introduced a new velocity of light which we feel could be used in lieu of the normal group velocity and the velocity of energy transport. This new definition has several advantages: It yields a unique value regardless of the dispersion present; its behavior in a region of absorption or gain is not erratic as the normally defined group velocity is; it eliminates the difficult problem of trying to derive a general relationship between the group velocity and the velocity of energy transport; since it is always measurable, the lack of measurability that troubled the conventional definition of v_e does not cause difficulty here.

It was found that one can measure experimentally the phase velocity, the group velocity, the centrovelocity, the relativistic velocity constant, and the ratio of units velocity. However, the signal velocity and the velocity of energy transport, when the standard definition is used, cannot be measured. However, if the centrovelocity is adopted for the description of the transport of energy, there is a measurable velocity which characterizes the flow of electromagnetic radiation energy.

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