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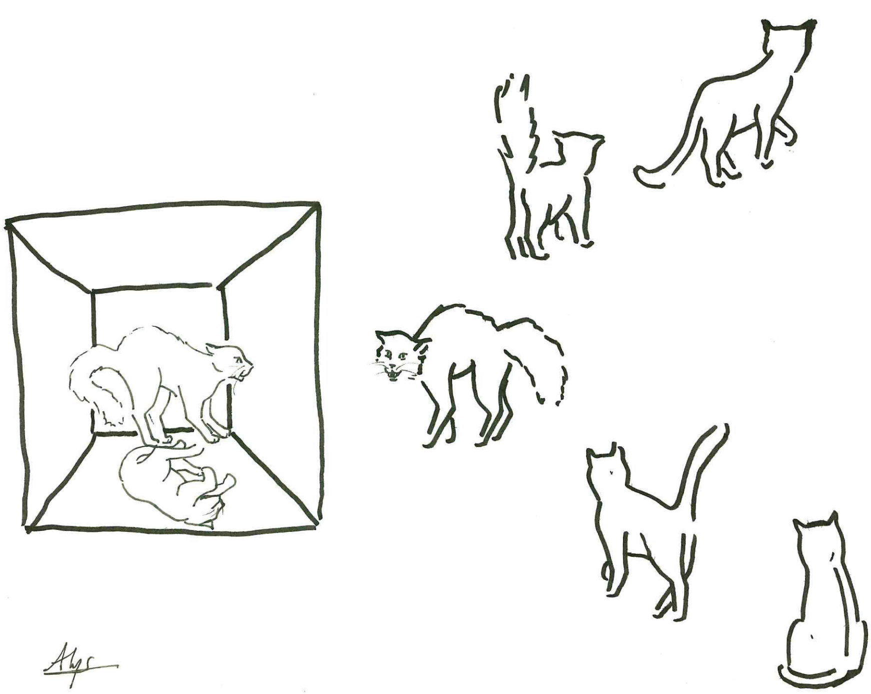
Émergence de dynamiques classiques en probabilités quantiques



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Ils ont bien failli gagner la bataille,
Pierre Perret

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Abstract

This thesis focuses on the study of several bridges between classical probability theory and quantum open systems theory.

The goal of the first part of this manuscript is to study the emergence of classical noises in the *quantum Langevin Equation.* This equation models the action of a quantum bath on a small system in the Markovian approximation. The discrete-time analogue of this equation, namely the *quantum repeated interaction* scheme, was developed by Stéphane Attal and Yan Pautrat. In previous works, Attal and his collaborators show that the natural noises that appear in this context are the *obtuses random variables* and study their structure. However, are they the only noises that can emerge and what to say about the general case? In the same way, it was more or less known that the only classical noises emerging in the quantum Langevin Equation are Brownian and Poisson processes. My contribution in this manuscript consists in defining a relevant von Neumann algebra on the environment, called the *Noise Algebra*, which encodes the structure of the noise. This algebra is commutative if and only if the noises are classical, where in this case we confirm the above mentioned statements on the nature of the noises. In the general case, with the help of this algebra, we can show a decomposition of the environment between a classical and a quantum part.

In the second part of the manuscript, we focus on classical stochastic processes that appear inside the quantum system. The dynamics of the system is still quantum, yet there exists an observable that evolves in a classical way. This is naturally the case when the quantum Markov semigroup leaves invariant a commutative maximal von Neumann subalgebra. We propose a method in order to generate such semigroups, based on a definition of Attal of a certain kind of dilations of classical Markov operators. We then show that *Lévy processes* have such quantum extensions.

In the second part again, we study an other kind of classical processes, induced by *Open Quantum Walks.* These walks, recently defined by Attal and his collaborators, generate a classical process (that is, classical trajectories) but with strong quantum behavior. We define in this context a Dirichlet problem, that we solve by a variational approach, using non commutative Dirichlet forms.

Finally, the last part is dedicated to the study of *Environment Induced Decoherence*. This fundamental notion in physics gives a dynamical explanation of the disappearance of Schrödinger's cat states in the classical world. We show that such a decoherence always occurs on finite von Neumann algebras under some natural assumptions. We then propose an innovative study of the speed of decoherence, based on non commutative functional inequalities. This approach highlights the role of entanglement in the decoherence process.

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Introduction

I Présentation et organisation du manuscrit

Cette thèse se consacre à l'étude de certaines passerelles reliant dynamiques classiques et quantiques dans la théorie des systèmes quantiques ouverts et s'inscrit dans le cadre des probabilités quantiques.

Les probabilités quantiques sont l'analogue non commutatif de la théorie kolmogorovienne des processus stochastiques. L'une des principales motivations est le développement d'une théorique mathématique solide des systèmes quantiques ouverts et de leur approche à l'équilibre, ou plus généralement de la physique statistique quantique. Les principaux objets avec lesquels j'ai travaillé sont les semi-groupes de Markov quantiques agissant sur des algèbres de von Neumann et le calcul stochastique non commutatif. Ils apportent ensemble une théorie mathématique cohérente pour décrire l'évolution irréversible de systèmes quantiques ouverts. Le comportement asymptotique de tels semi-groupes est intimement relié à l'approche à l'équilibre en physique statistique quantique. Le calcul stochastique quantique est ensuite utilisé pour intégrer dans cette description les degrés de liberté pertinent de l'environnement, responsables du comportement dissipatif. C'est dans ce cadre que s'inscrit le présent manuscrit.

L'ensemble des travaux présentés dans cette thèse portent sur l'apparition de processus stochastiques classiques dans l'approche markovienne de systèmes quantiques ouverts. Nous traitons cette problématique sur plusieurs niveaux :

Partie A : Bruits classiques : Dans l'approximation markovienne des systèmes quantiques ouverts, l'évolution de l'ensemble formé par un système et son bain est décrite par une famille d'opérateur unitaires, solution d'une équation stochastique quantique appelée *Équation de Langevin quantique*. La dynamique est ainsi modélisée comme l'équation d'un système fermé perturbée par des bruits quantiques. Le cas où ces bruits sont en réalité classiques est particulièrement important. C'est précisément l'objet de la partie A de cette thèse. Ma contribution principale dans ce cadre est la définition d'une algèbre de von Neumann sur l'environnement appelée *algèbre du bruit*. Elle permet en particulier de déterminer si l'action d'un bain quantique sur un petit système peut se ramener ou non à des bruits classiques. Nous reviendrons sur ce point dans la partie III de cette introduction. Comme conséquence de cette approche, nous obtenons une décomposition unique de toute dynamique quantique en une partie classique et une partie purement quantique. Dans le chapitre 1 je développe cette décomposition et des résultats qui lui sont liés en temps discret. La même approche est ensuite utilisée dans le chapitre 2 en temps continu.

Partie B : Réduction classique (ou extension quantique) : Dans certains cas, l'algèbre de von Neumann décrivant les observables du système contient une sous-algèbre commutative invariante et la restriction de la dynamique à cette sous-algèbre est un processus stochastique classique. On parle ainsi de réduction classique, ou bien d'extension quantique si on part du processus classique pour l'étendre en une dynamique quantique. Dans le chapitre 3, nous décrivons une "recette" qui permet dans certains cas de trouver une extension quantique d'un semi-groupe de Markov donné. Cette méthode permet entre autre de montrer que les processus de Lévy possèdent de telles extensions.
Dans cette même partie nous nous intéresserons aux *Marches Aléatoires Quantiques* introduites par Stéphane Attal et ses collaborateurs dans [APSS12]. Même si celles-ci n'entrent pas à proprement parler dans cette catégorie de dynamiques quantiques, elles s'en rapprochent grandement. En effet, elles correspondent à ces canaux quantiques sur des systèmes biparties qui laissent invariant une sous-algèbre commutative de l'une des parties seulement. Ce lien sera détaillé dans la partie **??** de cette introduction. Cette partie classique implique une certaine notion de trajectoire qui permet de décrire complètement l'évolution du système. Dans le chapitre 4, nous nous intéressons aux propriétés géométriques de ces trajectoires. Plus particulièrement, nous définissons un problème de Dirichlet associé aux marches quantiques ouvertes, que nous résolvons par une approche variationnelle. Nous obtenons au passage un résultat liant canaux quantiques et formes de Dirichlet non-commutatives.

Partie C : Décohérence : Il existe plusieurs définition de la décohérence en physique quantique. L'idée principale est proche des réductions classiques évoquées au dessus, où l'algèbre commutative est l'algèbre des opérateurs diagonaux dans une base orthonormée.

En plus de l'invariance de cette algèbre, la décohérence requiert que les termes hors-diagonaux disparaissent en temps long, le système étant donc asymptotiquement classique et décrit par une algèbre commutative. Ce phénomène revête une grande importance en physique, étant donné qu'il représente à la fois une limitation dans la conception d'ordinateurs quantiques - due à la perte des propriétés quantiques du système - mais aussi une ressource pour protéger le système d'effets dissipatifs. Dans la partie C de cette thèse, nous nous intéressons à une définition particulière de décohérence, adaptée aux évolutions markoviennes, appelée *Décohérence Induite par l'Environnement*, introduite par Blanchard et Olkiewicz dans [BO03]. Tout d'abord, nous prouvons sous une hypothèse naturelle que la décohérence à toujours lieu pour des systèmes modélisés par des algèbres de von Neumann finies. Nous abordons ensuite une étude innovante sur la vitesse de décohérence, s'appuyant sur de nouvelles inégalités fonctionnelles non-commutatives. Ce dernier point sera détaillé dans la partie V de cette introduction.

Tous les différents chapitres de cette thèse ont fait l'objet d'un article soumis à une revue international à comité de relecture, ou sont des articles en préparation :

- I. Bardet, *Quantum extensions of dynamical systems and of Markov semigroups*, soumis à Stochastic Analysis and Applications (cf chapter 3)
- I. Bardet, *Classical and Quantum Parts of the Quantum Dynamics : the discrete-time case*, accepté à Annales Henry Poincaré (cf chapter 1)
- S. Attal, I. Bardet, *Classical and Quantum Parts of the Environment for Quantum Langevin Equations*, soumis à Annales Henry Poincaré (cf chapter 2)
- I. Bardet, D. Bernard et Y. Pautrat *Exit times, return times and Dirichlet problems for open quantum walks*, en préparation (cf chapter 4)
- I. Bardet, *Environment-Induced-Decoherence for finite von Neumann algebra and speed of Decoherence*, en préparation (cf chapter 5)

La suite de cette introduction est dédiée à remettre dans un contexte plus large les résultats présentés dans cette thèse. Je me contenterai ainsi de ne citer que brièvement les résultats exposés dans le corps du manuscrit et me concentrerai davantage sur les motivations.

Dans la partie II nous introduisons donc de manière succincte le formalisme des probabilités non-commutatives et celui des dynamiques markoviennes des systèmes

quantiques ouverts.
Ensuite, dans la partie III, nous motivons la définition de l'algèbre du bruit ou algèbre de l'environnement, dont l'étude est le sujet central de la partie A de cette thèse.
Dans la partie **??** de cette introduction, nous tentons d'unifier les travaux de la partie B sur les dynamiques sous-classiques et les marches quantiques ouvertes.
Finalement, dans la partie V nous concluons cette introduction en présentant un exemple simple et pertinent de système quantique ouvert subissant une décohérence induite par l'environnement, illustrant la théorie développée dans le chapitre 5 sur la décohérence.

II Une courte introduction aux probabilités non-commutatives

Dans cette partie, nous présentons les différents concepts abordés dans cette thèse, tous en lien avec le domaine des mathématiques communément appelés «probabilités non commutatives». Le but principal étant d'introduire les principales notations de ce manuscrit, nous n'abordons cette vaste thématique que très brièvement.

Pour comprendre en quoi consiste les probabilités "non commutatives", expliquons tout d'abord en quoi on peut considérer les probabilités, au sens usuel du terme, comme "commutatives".
Prenons une variable aléatoire X définie sur son espace de probabilité canonique $(\Omega, \mathcal{F}, \mathbb{P})$. Il est certain que toute l'information nécessaire concernant X est contenue dans $\mathcal{F}$ et donc dans l'ensemble des fonctions $\mathcal{F}$-mesurables à valeurs complexes.
En conclusion, toute l'information est contenue dans l'algèbre $\mathcal{A}$ de fonctions essentiellement bornées[1] sur l'espace $(\Omega, \mathcal{F}, \mathbb{P})$:

$$\mathcal{A} = L^\infty(\Omega, \mathcal{F}, \mathbb{P}).$$

On peut de plus réaliser $\mathcal{A}$ comme une *algèbre de von Neumann commutative* sur l'espace de Hilbert $L^2(\Omega, \mathcal{F}, \mathbb{P})$ des fonctions de carrés intégrables sur $(\Omega, \mathcal{F}, \mathbb{P})$. Cela se fait naturellement en identifiant un élément $f \in \mathcal{A}$ par l'opérateur de multiplication M_f défini comme :

$$\begin{array}{cccc} M_f & : & L^2(\Omega, \mathcal{F}, \mathbb{P}) & \to & L^2(\Omega, \mathcal{F}, \mathbb{P}). \\ & & g & \mapsto & fg \end{array}$$

[1]rappelons que ess-sup f est le plus petit réel positif α tel que l'ensemble $\{|f| > \alpha\}$ soit de mesure nulle

De plus la mesure de probabilité $\mathbb{P}$ induit sur $\mathcal{A}$ une fonctionnelle ω à valeurs complexes par intégration :

$$\omega(f) = \int_\Omega f d\mathbb{P}.$$

C'est cette reformulation algébrique des probabilités que l'on va généraliser dans un contexte non-commutatif. Dans la sous-partie suivante II. 1 nous définissons et donnons les principaux exemples d'espace de probabilités non commutatifs. Nous rappelons ensuite dans la sous-partie II. 2 le formalisme mathématique des dynamiques quantiques markoviennes.

II. 1 Espace de probabilité non-commutatif

Rappelons qu'une *Algèbre de von Neumann* $\mathcal{M}$ sur un espace de Hilbert $\mathcal{H}$ est une $*$-algèbre de l'espace de Banach $\mathcal{B}(\mathcal{H})$ des opérateurs bornés sur $\mathcal{H}$, qui de plus contient l'opérateur identité $I_\mathcal{H}$ et qui est fermée pour la topologie faible. Par le Théorème du bicommutant de von Neumann, c'est équivalent au fait que $\mathcal{M}$ est égal à son bicommutant : $\mathcal{M}'' = \mathcal{M}$, où le commutant d'un sous-ensemble S de $\mathcal{B}(\mathcal{H})$ est défini comme $S' = \{Y \in \mathcal{B}(\mathcal{H});\ [X,Y] = 0\ \forall X \in S\}$. Cette propriété garantit en particulier que $\mathcal{M}$ est engendrée par les projections orthogonales qu'elle contient, ce qui dans le cas commutatif revient à dire que l'espace des fonctions essentiellement bornées est engendré par les fonctions indicatrices.

Un *État* sur $\mathcal{M}$ est une application linéaire ω de $\mathcal{M}$ dans $\mathbb{C}$, qui respecte la positivité : $\omega(X^*X \leq 0\ \forall X \in \mathcal{M}$, et qui est normalisé : $\omega(I_\mathcal{H}) = 1$. Cet état est dit *normal* s'il est σ-faiblement continu, et *fidèle* si $\omega(X^*X) = 0$ implique $X = 0$.

En conclusion, la fermeture faible de $\mathcal{M}$ garantit que la théorie est complètement déterminée par l'ensemble des événements ; la normalité de l'état est équivalente à l'additivité dénombrable.

On arrive ainsi à la définition suivante.

Définition II.1. *On définie un* espace de probabilités non-commutatif *comme un triplet* $(\mathcal{M},\mathcal{H},\omega)$ *où* $\mathcal{M}$ *est une algèbre de von Neumann agissant sur l'espace de Hilbert* $\mathcal{H}$ *et* ω *est un état sur* $\mathcal{M}$*.*

Donnons maintenant les exemples principaux d'espace de probabilités non-commutatifs que nous allons rencontrer dans cette thèse.

Exemples II.1 (Algèbre des opérateurs bornés)**.** L'exemple le plus simple d'algèbre de von Neumann est bien sûr $\mathcal{B}(\mathcal{H})$. C'est le cas que nous allons considérer le plus souvent dans cette thèse pour modéliser l'espace des états purs d'un système quantique.

Nous notons $\mathcal{L}_1(\mathcal{H})$ l'ensemble des opérateurs à trace sur $\mathcal{H}$, qui s'identifie avec le *prédual* de $\mathcal{B}(\mathcal{H})$. Ainsi, pour chaque état normal ω sur $\mathcal{B}(\mathcal{H})$, il existe un unique opérateur à trace $\rho \in \mathcal{L}_1(\mathcal{H})$ tel quel

$$\omega(X) = \mathrm{Tr}[\rho X] \qquad \forall X \in \mathcal{B}(\mathcal{H}).$$

Positivité et normalisation de ω impose que ρ est un opérateur positif, de trace égale à 1. Un tel opérateur est appelé *matrice densité*. Dans la suite nous ferons souvent l'amalgame entre l'état ω et sa matrice densité ρ, en appelant notamment ρ un état sur $\mathcal{B}(\mathcal{H})$.

Exemples II.2 (Algèbre de von Neumann commutative)**.** Soit $\mathcal{A}$ une algèbre de von Neumann commutative sur $\mathcal{H}$ et ω un état normal sur $\mathcal{A}$. D'après le théorème spectral [KR97], il existe un espace de probabilité $(\Omega, \mathcal{F}, \mathbb{P})$ tel que $\mathcal{A}$ est $*$-isomorphe à $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$. De plus, en notant i cet isomorphisme, on a pour tout $f \in \mathcal{A}$:

$$\omega(f) = \int_\Omega i(f)\, d\mathbb{P}.$$

Soit M_f l'opérateur de multiplication par une fonction $f \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$. Remarquons que le $*$-homomorphisme $f \mapsto M_f$ contient lui aussi toute l'information sur la loi de X. En effet, $\langle \mathbb{1}, M_f \mathbb{1} \rangle = \mathbb{E}[f(X)]$, où $\mathbb{1}$ est la fonction indicatrice de l'ensemble Ω et on peut donc calculer grâce aux opérateurs M_f les moments, la fonction caractéristique, etc... qui caractérisent complètement la loi de probabilité $\mathbb{P}$.

Pour cette raison, une *variable aléatoire quantique* fut définie dans l'article fondateur d'Accardi, Frigerio et Lewis comme un $*$-homomorphisme unitaire entre une algèbre de von Neumann $\mathcal{M}$ et $\mathcal{B}(\mathcal{H})$ pour un certain espace de Hilbert $\mathcal{H}$ [AFL82].

Exemples II.3 (Algèbre de von Neumann finie)**.** Un type particulier d'algèbres de von Neumann que l'on rencontrera dans le dernier chapitre de cette thèse est celui des *algèbres de von Neumann finies*. Sans donner la définition exacte en terme de projecteurs orthogonaux, mentionnons juste que de manière équivalente une algèbre de von Neumann est finie si elle possède une trace finie que l'on note Tr, c'est à dire un état normal fidèle tracial :

$$\mathrm{Tr}[XY] = \mathrm{Tr}[YX], \qquad \forall X, Y \in \mathcal{B}(\mathcal{H}).$$

C'est donc en particulier le cas de $\mathcal{B}(\mathcal{H})$ si $\mathcal{H}$ est de dimension finie.

Une autre classe importante d'algèbre finie est celles des *algèbres de type II_1*, dont la trace peut prendre toutes les valeurs possibles de l'intervalle $[0,1]$. Physiquement, $\mathcal{B}(\mathcal{H})$

modélise des systèmes avec un nombre fini de degrés de libertés, tandis que les algèbres de type II_1 servent à modéliser des systèmes de spin infinis.

II. 2 Dynamique d'un système quantique ouvert

Nous allons maintenant décrire le formalisme mathématique des systèmes quantiques ouverts *dans l'approximation markovienne*. L'un des postulats de la mécanique quantique est que l'ensemble des observables d'un système est décrit par une algèbre de von Neumann $\mathcal{M}$, et que l'état du système est donné par un état normal σ sur $\mathcal{M}$.

Lorsque le système est *fermé*, c'est à dire quand il n'a aucun échange d'aucune sorte avec l'extérieur, son évolution dans le point de vue de Heisenberg est donné par un groupe à un paramètre, $*$-faiblement continu, de $*$-isomorphismes d'algèbres $(\alpha_t)_{t\in\mathbb{R}}$. Ainsi, à l'instant $t \geq 0$ une observable $X \in \mathcal{M}$ aura évolué comme,

$$X \mapsto \alpha_t(X). \tag{II.1}$$

Lorsque le système est *ouvert*, sa dynamique est décrite par un semi-groupe de Markov quantique (SMQ), c'est à dire un semi-groupe d'opérateurs linéaire $(\mathcal{P}_t)_{t\geq 0}$ sur $\mathcal{M}$, tel que :

- pour tout $t \geq 0$, l'application $\mathcal{P}_t$ est une application complètement positive et normale (c'est à dire continue pour la topologie σ-faible).
- $t \mapsto \mathcal{P}_t(X)$ est $*$-faiblement continue pour tout $X \in \mathcal{M}$.
- $\mathcal{P}$ est conservatif, c'est à dire $\mathcal{P}_t(I_{\mathcal{H}}) = I_{\mathcal{H}}$ pour tout $t \geq 0$.

Rappelons qu'une application complètement positive $\mathcal{L}$ sur $\mathcal{M}$ (ACP) est une application linéaire telle que ses extensions $\mathcal{L} \otimes I_n$ sur l'algèbre $\mathcal{M} \otimes \mathcal{M}_n(\mathbb{C})$ préservent la positivité pour tout $n \geq 0$. Le Théorème de Stinespring donne une très belle caractérisation des ACP.

Théorème II.1. *Une application linéaire $\mathcal{L}$ sur $\mathcal{M}$ est complètement positive si et seulement si elle est de la forme :*

$$\mathcal{L}(X) = V^* \pi(X) V, \qquad X \in \mathcal{M}, \tag{II.2}$$

où $(\pi, \mathcal{K})$ est une représentation de $\mathcal{M}$ sur l'espace de Hilbert $\mathcal{K}$ et où V est un opérateur borné de $\mathcal{H}$ dans $\mathcal{K}$.

Si de plus $\mathcal{L}$ est normale, alors on peut choisir $\mathcal{K}$ de la forme $\mathcal{H} \otimes \mathcal{K}'$ et π de la forme $X \mapsto X \otimes I_{\mathcal{K}'}$.

Exemples II.4 (Algèbre des opérateurs bornés)**.** Considérons le cas $\mathcal{M} = \mathcal{B}(\mathcal{H})$. Le groupe de $*$-automorphisme $(\alpha_t)_{t\in\mathbb{R}}$ prend alors la forme suivante :

$$X \mapsto U_t^* X U_t. \tag{II.3}$$

où $(U_t)_{t\geq 0}$ est un groupe d'opérateur unitaire fortement continu sur $\mathcal{H}$. De manière équivalente, l'évolution peut être écrite dans le point de vue de Schrödinger. Si l'état initial du système est donné par la matrice densité ρ, alors l'état à l'instant $t \geq 0$ est :

$$\rho \mapsto U_t \rho U_t^*. \tag{II.4}$$

En appliquant le Théorème de Stinespring à un SMQ $(\mathcal{P}_t)_{t\geq 0}$, on obtient après un peu de travail l'expression suivante :

$$\mathcal{P}_t(X) = \mathrm{Tr}_\omega\big[U_t^* \,(X \otimes I_{\mathcal{K}})\, U_t\big]. \tag{II.5}$$

où $(U_t)_{t\in\mathbb{R}}$ est un groupe unitaire sur l'espace de Hilbert $\mathcal{H} \otimes \mathcal{K}$ et où Tr_ω dénote la trace partielle par rapport à ω, c'est à dire l'unique application linéaire de $\mathcal{B}(\mathcal{H}\otimes\mathcal{K})$ dans $\mathcal{B}(\mathcal{H})$ telle que pour tout $X \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ et tout opérateur à trace ρ sur $\mathcal{H}$:

$$\mathrm{Tr}\big[\rho \mathrm{Tr}_\omega[X]\big] = \mathrm{Tr}\big[\,(\rho \otimes \omega)\, X\big].$$

L'Équation (II.5) a la signification suivante : l'espace de Hilbert $\mathcal{K}$ représente l'environnement du système, c'est à dire les degrés de liberté manquant pour décrire le système comme un système fermé. L'espace $\mathcal{H} \otimes \mathcal{K}$ est donc un système fermé qui lui évolue selon l'Équation de Heisenberg (II.3). L'évolution des observables sur le petit système uniquement est alors la trace de l'évolution sur le gros système.
Dans la partie A de cette thèse, nous focalisons notre attention sur le groupe unitaire $(U_t)_{t\geq 0}$. Insistons sur le fait que celui-ci n'est pas défini de manière unique.

Dans le point de vue de Schrödinger, l'évolution est décrite par le prédual $\mathcal{P}_*$ du SMQ qui prend la forme suivante :

$$\mathcal{P}_{*t}(\rho) = \mathrm{Tr}_{\mathcal{K}}\big[U_t\,(\rho \otimes \omega)\, U_t^*\big], \tag{II.6}$$

où cette fois $\mathrm{Tr}_{\mathcal{K}}$ est la trace partielle par rapport à l'espace $\mathcal{K}$, unique application linaire de l'ensemble des opérateurs à trace $\mathcal{L}_1(\mathcal{H} \otimes \mathcal{K})$ sur $\mathcal{H} \otimes \mathcal{K}$ sur $\mathcal{L}_1(\mathcal{H})$ telle que pour tout $\rho \in \mathcal{L}_1(\mathcal{H} \otimes \mathcal{K})$ et tout $X \in \mathcal{B}(\mathcal{H})$,

$$\mathrm{Tr}\big[X \mathrm{Tr}_{\mathcal{K}}[\rho]\big] = \mathrm{Tr}\big[\,(X \otimes I_{\mathcal{K}})\, \rho\big].$$

Exemples II.5 (Algèbre commutative)**.** Dans le chapitre 3 nous serons particulièrement intéressés par le cas d'une algèbre de von Neumann commutative $\mathcal{A}$. Les notions précédentes se résument alors aux notions classiques. Rappelons que si $\mathcal{M}$ est commutative, étant donné un état normal ω sur $\mathcal{M}$, elle s'identifie à l'espace $L^\infty(\Omega, \mathcal{F}, \mathcal{P})$ où $(\Omega, \mathcal{F}, \mathcal{P})$ est un espace de probabilité.
L'équation (II.1) implique l'existence d'un groupe à un paramètre continu $(T_t)_{t\in\mathbb{R}}$ d'applications sur Ω telles que pour tout $t \in \mathbb{R}$ et tout $f \in L^\infty(\Omega, \mathcal{F}, \mathcal{P})$, on ait

$$\alpha_t(f) = f \circ T_t.$$

Le semi-groupe de Markov se résume quant à lui un semi-groupe de Markov au sens classique du terme.

III Présentation de la partie I : Parties classiques et quantiques de l'environnement

Dans cette partie, l'espace de probabilité quantique que nous considérons est $\mathcal{B}(\mathcal{H})$ pour un certain espace de Hilbert $\mathcal{H}$. Par simplicité, nous supposons que $\mathcal{H}$ est de dimension finie N, la plupart des résultats que nous présentons étant encore vrais lorsque $\mathcal{H}$ est de dimension infinie.

Notre but est de discuter et surtout de motiver les résultats de la partie A de ce manuscrit, qui portent sur l'apparition de bruits classiques dans l'environnement d'un système quantique ouvert. Notez que cette discussion se veut avant tout informelle.

Nous avons vu dans la partie précédente que lorsque le système est fermé, celui-ci évolue selon un groupe unitaire $(U_t)_{t\geq 0}$. Vu que $\mathcal{H}$ est de dimension finie, il existe un opérateur autoadjoint $H \in \mathcal{B}(\mathcal{H})$ tel que $(U_t)_{t\geq 0}$ est solution de l'équation suivante, dite *Équation de Schrödinger*.

$$\begin{cases} dU_t = -iHU_t dt & \forall t \geq 0, \\ U_0 = I_{\mathcal{H}}. \end{cases}$$

Une question que l'on peut se poser dans l'étude des systèmes hors-équilibre est comment modéliser un bain thermique, ou plus simplement comment modéliser l'action de l'environnement responsable de la dissipation. Il est alors naturel de chercher à modéliser l'action de l'environnement par l'action d'un bruit stochastique ayant pour effet de perturber l'équation de Schrödinger précédente.

Donnons un exemple en temps discret. Prenons un processus stochastique $(X_n)_{n\geq 0}$ à valeur dans $\mathbb{R}^d$, défini sur un espace de probabilités $(\Omega, \mathcal{F}, \mathcal{P})$. Notons $(X_n^1, ..., X_n^d)$ ses coordonnées au pas de temps n. On peut alors tenter de donner un sens à l'équation suivante, où $A, B_1, ..., B_d \in \mathcal{B}(\mathcal{H})$:

$$\begin{cases} U_{n+1} - U_n = AU_n + \sum_{k=1}^d B_k U_n X_n^k & \forall n \geq 0, \\ U_0 = I_{\mathcal{H}}. \end{cases} \tag{III.1}$$

Le processus $(X_n)_{n\geq 0}$ et les coefficients $A, B_1, ..., B_d$ obéissent à certaines contraintes : il faut en particulier que presque sûrement U_n soit un opérateur unitaire sur $\mathcal{H}$, pour tout $n \geq 0$.

On peut aussi considérer la situation analogue en temps continu. Soit $(X_t)_{t\geq 0}$ un processus stochastique à valeurs dans R^d, de coordonnées $X_t^1, ..., X_t^d$. L'équation de Schrödinger perturbée par ce bruit donne :

$$\begin{cases} dU_t = AU_t \; dt + \sum_{k=1}^d B_k U_t \; dX_t^k & \forall t \geq 0, \\ U_0 = I_{\mathcal{H}}. \end{cases} \tag{III.2}$$

Encore une fois, le processus ainsi que les coefficients ne peuvent être quelconque. En particulier, le processus doit être tel qu'il est existe une théorie d'intégration par rapport à ses accroissements (typiquement une martingale semi-continue). Cependant, on peut s'attendre à ce que pour certains choix, la solution soit un processus aléatoire à valeurs dans le groupe unitaire de $\mathcal{H}$.

Il est bien sûr évident que cette manière de modéliser l'environnement n'inclut pas toutes les dynamiques décrites par des semi-groupes de Markov quantiques. En effet, quand elles sont bien définies, les solutions des équations (III.1) et (III.2) donnent, en prenant l'espérance, des semi-groupes de Markov quantiques. Par exemple, dans le cas du temps continu :

$$\mathcal{P}_t(X) = \mathbb{E}[U_t^* X U_t] \qquad t \geq 0.$$

Dans un sens, le processus stochastique $(U_t)_{t\geq 0}$ est une *dilatation* de $\mathcal{P}$.
Il n'en est pas moins intéressant de tacher de comprendre quels semi-groupes admettent de telles dilatations. Ou autrement dit : *quand peut-on dire que l'environnement a une action classique sur le système.*

L'approche que l'on aborde dans cette thèse, amorcée par Stéphane Attal et ses collaborateurs, est la suivante. On part d'une famille d'unitaire sur un «gros» espace, représentant le système et son environnement, solution d'une certaine équation. On cherche alors à ne faire apparaître l'environnement dans cette équation que sous la forme d'un bruit classique. Plus rigoureusement, l'environnement se manifestera comme l'opérateur de multiplication par un processus stochastique.

Je suis volontairement resté très vague sur le modèle utilisé dans ce cadre pour modéliser l'environnement, en particulier sur l'espace de probabilité non-commutatif le modélisant. Il faut ainsi nous restreindre à un certain type de dilatations de semi-groupes de Markov quantiques. La théorie des dilatations étant très vaste, on se restreint au choix le plus concret et le plus utilisé. Dans le cas du temps discret, le cadre le plus adapté est celui des interactions répétées développé par Stéphane Attal et Yan Pautrat. Dans le cas du temps continu, on se place dans le cadre des Équations de Hudson et Parthasarathy ou Équation de Langevin quantique [HP84]. Ces deux cas sont particulièrement adaptés puisqu'ils permettent de dilater de manière explicite toute dynamique Markovienne.

Une fois que le cadre sera bien posé, nous pourrons répondre aux questions suivantes :

1. Quels bruits classiques sont admissibles, c'est à dire peuvent effectivement apparaître dans les équations quantiques ?
2. Quels sont alors les contraintes sur les coefficients de l'équation pour qu'elle admette comme unique solution une famille d'unitaires ?
3. Dans le cas général, existe-t-il des parties de l'environnement qui agissent de manière classique sur le système et si oui comment les identifier ?

En réalité, les réponses aux deux premières questions étaient déjà connus. Pour la deuxième, il suffit d'appliquer les conditions propres à l'équation considérée : c'est le choix du type de dilatation qui impose donc ces contraintes. Pour la première, dans le cas continu, on sait depuis l'article original qu'il est possible de faire apparaître le mouvement brownien et les processus de Poisson compensés dans l'Équation de Langevin quantique. Dans le cas discret [HP84], Attal et ses collaborateurs ont montré comment certaines approximations discrètes de ces deux processus peuvent apparaître dans le schéma des interactions répétées [ADP]. Dans les deux cas, il n'existait aucune preuve attestant que c'étaient les seules possibilités.

Ma contribution est la définition d'une certaine algèbre de von Neumann sur l'environnement, qui contient toute l'information sur la nature du bruit. Ainsi, dans le cas où elle est commutative, elle correspond exactement à celle engendrée par un processus stochastique classique. Il est alors possible d'identifier la nature de ce processus.
Dans le cas général, il est aussi possible d'identifier grâce à cette algèbre quelles sont les parties classiques de l'environnement et ainsi de répondre à la troisième question.

Dans la suite de cette partie, nous détaillons les deux types de bains, en temps discret et continu, que nous avons utilisés dans notre étude. Nous résumons aussi les résultats des chapitres 1 et 2.

III. 1 Interactions quantiques répétées

Dans cette sous-partie nous présentons le modèle de bain quantique permettant de dilater des semi-groupes de Markov quantique à temps discret. Ce modèle, appelé *Interactions Quantiques Répétées*, fut introduit par Stéphane Attal et Yan Pautrat dans une série d'articles culminant dans [AP06]. L'environnement est modélisé par une chaîne de sous-systèmes identiques, chacun venant interagir de manière successive avec le système. On note $\mathcal{K}$ l'espace de Hilbert représentant l'espace des états d'une particule. L'espace modélisant une chaîne dénombrable de particules identiques et indépendantes est alors :

$$T\Phi = \bigotimes_{n\geq 0} \mathcal{K}_n$$

Supposons que l'évolution pour un pas de temps soit donnée par un opérateur unitaire U sur $\mathcal{H}\otimes\mathcal{K}$. On fait de plus l'hypothèse que l'évolution est homogène en temps, de telle sorte que l'opérateur U est le même pour chaque pas de l'évolution. Cela nous donne un opérateur U_n sur $T\Phi$ décrivant le n-ième pas de l'évolution :

$$U_n = I_{\mathcal{K}_0} \otimes \cdots \otimes I_{\mathcal{K}_{n-1}} \otimes U \otimes I_{\mathcal{K}_{n+1}} \otimes \cdots,$$

où U agit sur $\mathcal{H}\otimes\mathcal{K}_n$. L'opérateur unitaire décrivant l'évolution jusqu'au temps n est alors la solution de l'*Équation aux interactions répétées*

$$\begin{aligned} V_0 &= I, \\ V_{n+1} &= U_{n+1} V_n. \end{aligned} \tag{III.3}$$

Rappelons que notre but est de comprendre quand, dans ce cadre, il est possible d'affirmer que l'environnement est classique. En particulier, cela doit se manifester par le fait que l'équation (III.3) se réécrive avec des bruits classiques.
L'idée majeure apportée par cette thèse consiste à donner un sens intrinsèque au caractère classique de l'environnement $T\Phi$, à travers la définition d'une algèbre de von Neumann pertinente sur $T\Phi$. On pourra alors dire que l'environnement est *classique* lorsque cette algèbre est commutative.

Définition III.1. *Soit* $(V_n)_{n\geq 0}$ *la solution de l'équation* (III.3). *On définit l'*algèbre du bruit $\mathcal{A}_n(V)$ *au temps* $n \in \mathbb{N}$ *comme la plus petite algèbre de von Neumann sur* $T\Phi$ *telle* *que*

$$V_k \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{A}_n(V) \text{ pour tout } k \leq n.$$

De manière équivalente, $\mathcal{A}_n(V)$ est définie par son commutant :

$$\mathcal{A}_n(V)' = \{Y \in \mathcal{B}(T\Phi);\ [I_{\mathcal{H}} \otimes Y, V_k] = [I_{\mathcal{H}} \otimes Y, V_k^*] = 0 \quad \forall k \leq n\}.$$

Comme on a l'inclusion canonique $\mathcal{A}_n(V) \subset \mathcal{A}_{n+1}(V)$ pour tout $n \geq 0$, il est possible de définir la limite inductive de cette famille d'algèbres, que l'on note $\mathcal{A}_\infty(V)$.

Cette définition ne sera pertinente que si, dans le cas où cette algèbre est commutative, l'équation (III.3) peut être réécrite avec des bruits classiques. Ceci est l'objet du théorème suivant, dont une partie est prouvée dans l'article de Attal, Deschamps et Pellegrini [ADP].

Théorème III.1. *Les assertions suivantes sont équivalentes :*

1. *$\mathcal{A}_\infty(V)$ est l'algèbre canonique d'une marche obtuse ;*
2. *$\mathcal{A}_n(V)$ est commutative pour tout $n \geq 0$;*
3. *V_n est solution de l'équation stochastique à temps discret*

$$\begin{cases} U_{n+1} - U_n = AU_n + \sum_{k=1}^d B_k U_n M_{X_n^k} & \forall n \geq 0, \\ U_0 = I_{\mathcal{H}}. \end{cases} \tag{III.4}$$

où $A, B_1, \dots, B_d \in \mathcal{B}(\mathcal{H})$ et où $M_{X_n^k}$ est l'opérateur de multiplication par la k-ième coordonnée d'une variable obtuse au temps n.

Les variables obtuses forment une classe particulière de variables aléatoires à valeurs dans $\mathbb{R}^d$. Elles furent introduites par Attal et Émery dans [AÉ94], puis plus tard étudiées dans le contexte des interactions quantique répétées par Attal et Pautrat [AP05] (voir aussi [ADP13] pour le cas complexe). Comme nous n'allons pas en avoir besoin dans la suite, je préfère ne pas les introduire ici. Mentionnons tout de même le caractère générique des variables obtuses : elles génèrent dans un certain sens toutes les variables centrées et réduites à valeurs dans $\mathbb{R}^d$ et ne prenant qu'un nombre fini de valeurs.

On peut déduire de ce théorème que, dans le cadre des interactions répétées, les seuls bruits admissibles sont les marches obtuses. En effet, supposons que $(\tilde{V}_n)_{n\geq}$ soit solution d'une équation stochastique du type Équation (III.1), où $(X_n)_{n\geq 0}$ est un processus stochastique homogène à valeurs dans $\mathbb{R}^d$ centré et réduit et ne prenant qu'un nombre fini de valeurs. Remarquons tout d'abord que cette dernière restriction est nécessaire pour que l'espace de Hilbert canonique associé à un pas de l'évolution soit de dimension finie. Notons $\mathcal{K}$ cet espace. L'espace de Hilbert canonique associé au processus $(X_n)_{n\geq 0}$ est donc $T\Phi$. On peut alors remplacer le processus $(X_n^k)_{n\geq 0}$ dans l'équation par son opérateur de multiplication $M_{X_n^k}$, agissant sur la k-ième copie de $\mathcal{K}$. On obtient ainsi une équation du type (III.3), admettant comme unique solution une famille d'unitaire $(V_n)_{n\geq 0}$. Le caractère générique des variables obtuses permet ensuite d'affirmer que l'algèbre du bruit $\mathcal{A}_\infty(V)$ est isomorphe à celle engendrée par une variable obtuse.

Nous avons ainsi montré que toute l'information sur la nature stochastique du processus $(V_n)_{n\geq 0}$ est contenue dans l'algèbre du bruit $\mathcal{A}_\infty(V)$. Partant d'une équation stochastique III.1, nous avons réécrit le problème de manière déterministe en remplaçant le bruit par l'opérateur de multiplication associé. Toute la nature stochastique du bruit est alors contenue dans l'algèbre du bruit $\mathcal{A}_\infty(V)$.
De plus, le processus étant homogène en temps, on peut se ramener à l'étude de l'algèbre $\mathcal{A}(U)$ défini pour un pas d'évolution. Cette algèbre est le sujet principal de l'article [Bar15], présenté dans la sous-partie suivante.

III. 2 Présentation de l'article : Parties classiques et quantiques d'une dynamique quantique : le cas du temps discret

Dans l'article [Bar15], j'étudie l'algèbre du bruit $\mathcal{A}(U)$ pour un pas d'évolution. Le contexte est donc le suivant : $\mathcal{H}$ et $\mathcal{K}$ sont deux espaces de Hilbert, modélisant respectivement le système et son environnement et U est un opérateur unitaire sur $\mathcal{H}\otimes\mathcal{K}$. Dans ce contexte, j'appelle l'*algèbre de l'environnement* l'algèbre $\mathcal{A}(U)$ dont je rappelle la caractérisation en terme de son commutant :

$$\mathcal{A}(U)' = \{Y \in \mathcal{B}(\mathcal{K});\ [I_\mathcal{H}\otimes Y, U] = [I_\mathcal{H}\otimes Y, U^*] = 0\}.$$

On peut montrer que si $\mathcal{A}(U)$ est commutative et $\mathcal{K}$ est de dimension finie d, alors il existe une base orthonormée $(\psi_i)_{i=1}^d$ de $\mathcal{K}$ et des opérateurs unitaires $U_1, ..., U_d$ sur $\mathcal{H}$ tels que

$$U = \sum_{i=1}^{d} U_i \otimes |\psi_i\rangle\langle\psi_i|. \tag{III.5}$$

Cette forme d'unitaire correspond à la définition initiale de Attal et al. d'un *environnement classique*. Ainsi, l'algèbre de l'environnement permet d'en donner une définition plus intrinsèque, qui de plus se généralise en dimension quelconque.

Décomposition de l'environnement

La première partie de l'article se concentre sur la décomposition de l'environnement entre une partie classique et une partie quantique. Pour fixer les idées, illustrons nos propos avec l'exemple suivant :

Exemples III.1. On considère l'opérateur unitaire U_{ex} sur $\mathbb{C}^2 \otimes \mathbb{C}^4$, écrit dans la base orthonormée canonique (e_1, e_2, e_3, e_4) de $\mathcal{K} = \mathbb{C}^4$ $(\alpha, \beta \in \mathbb{R},\ \alpha \neq \beta)$:

$$U_{\text{ex}} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos\beta & -\sin\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & \sin\beta & \cos\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{III.6}$$

Clairement l'environnement est la somme directe de deux sous-espaces, $\mathcal{K}_c = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ et $\mathcal{K}_q = \mathbb{C}e_3 \oplus \mathbb{C}e_4$, tels que $\mathcal{H} \otimes \mathcal{K}_{c,q}$ sont stables par U_{ex} et :

- La restriction U_c de U sur $\mathcal{H} \otimes \mathcal{K}_c$ a, d'après le sens donné dans la sous-partie précédente, un environnement classique. En effet :

$$\begin{aligned} U_c &= \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & \cos\beta & -\sin\beta \\ 0 & 0 & \sin\beta & \cos\beta \end{pmatrix} \\ &= \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \otimes |e_1\rangle\langle e_1| + \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \otimes |e_2\rangle\langle e_2|, \end{aligned}$$

  de sorte que $\mathcal{A}(U_c) = \mathbb{C}|e_1\rangle\langle e_1| + \mathbb{C}|e_2\rangle\langle e_2| = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix},\ a, b \in \mathbb{C} \right\}$ as $\alpha \neq \beta$.

- La restriction U_q de U_{ex} sur $\mathcal{H} \otimes \mathcal{K}_q$ a un environnement quantique. En effet, comme on le montre dans l'article, $\mathcal{A}(U_q) = \mathcal{B}(\mathcal{K}_q)$.

Ainsi, dans cet exemple, $\mathcal{K}_c$ est un sous-espace commutatif de l'environnement, ce qui se caractérise par le fait que $\mathcal{H} \otimes \mathcal{K}_c$ réduit à la fois U et U^*, et que $\mathcal{A}(U_c)$ est commutative. De plus, toutes les autres parties non triviales de l'environnement qui pourraient être considérer comme classiques, sont des sous-espaces de $\mathcal{K}_c$.

Cet exemple nous conduit à la définition suivante de "sous-partie" classique de l'environnement.

Définition III.2. *Soit U un opérateur unitaire sur $\mathcal{H}\otimes\mathcal{K}$. Soit $\tilde{\mathcal{K}}$ un sous-espace de $\mathcal{K}$. On dit que $\tilde{\mathcal{K}}$ est un* sous-espace commutatif de l'environnement *si $\tilde{\mathcal{K}} \neq \{0\}$ et :*

- *$\mathcal{H}\otimes\tilde{\mathcal{K}}$ et $\mathcal{H}\otimes\tilde{\mathcal{K}}^{\perp}$ sont stables par U,*
- *$\mathcal{A}(\tilde{U})$ est commutative, où $\tilde{U}$ est la restriction de U à $\mathcal{H}\otimes\tilde{\mathcal{K}}$.*

On voit bien ici le rôle central joué par l'algèbre $\mathcal{A}(U)$. Nous obtenons à partir de cette définition une décomposition de l'environnement entre une partie classique et une partie quantique

Théorème III.2. *L'espace de Hilbert $\mathcal{K}$ est la somme directe de deux sous-espaces $\mathcal{K}_c$ et $\mathcal{K}_q$, tels que soit $\mathcal{K}_c = \{0\}$, soit :*

- *$\mathcal{K}_c$ est un sous-espace commutatif de l'environnement.*
- *Si $\tilde{\mathcal{K}}$ est un sous-espace commutatif de l'environnement, alors $\tilde{\mathcal{K}}$ est un sous-espace de $\mathcal{K}_c$.*
- *La restriction de U à $\mathcal{H}\otimes\mathcal{K}_q$ n'a aucun sous-espace commutatif.*

Action à droite de l'environnement

L'unitaire défini par l'équation (III.5) à la particularité suivante. Pour toute matrice densité ω sur $\mathcal{K}$, l'application complètement positive $\mathcal{L}_\omega$ définie ci-dessus est un canal quantique *unitaire aléatoire* :

$$\mathcal{L}_\omega(X) = \mathrm{Tr}_\omega[U^*(X\otimes I_{\mathcal{K}})U] = \sum_{i=1}^{d} p_i U_i^* X U_i,$$

où $p_i = \langle \psi_i, \omega\, \psi_i\rangle$. Attal et al. remarquent alors qu'en modifiant légèrement la définition de U, on obtient exactement le même canal quantique. Ainsi, si $(\varphi_i)_{i=1}^d$ est une autre base orthonormée et si on définit l'opérateur unitaire V sur $\mathcal{H}\otimes\mathcal{K}$:

$$V = \sum_{i=1}^{d} U_i \otimes |\varphi_i\rangle\langle\psi_i|, \tag{III.7}$$

alors V et U définissent la même application complètement positive $\mathcal{L}_\omega$ pour toute matrice densité ω. Pourtant $\mathcal{A}(V)$ n'a que peu de chance en toute généralité d'être commutative. Cela montre que la notion d'algèbre de l'environnement n'est pas la bonne lorsque l'on s'intéresse à l'évolution des observables du systèmes seulement. Pour pallier à ce problème j'introduis une autre algèbre, l'algèbre de l'action à droite.

Définition III.3. *On définit l'*algèbre de l'action à droite de l'environnement $\mathcal{A}_r(U)$ *la plus petite algèbre de von Neumann sur* $\mathcal{K}$ *telle que pour tout* $X \in \mathcal{B}(\mathcal{H})$,

$$U^*(X \otimes I_{\mathcal{K}})U \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{A}_r(U).$$

De manière équivalente, cette algèbre est définie par son commutant :

$$\mathcal{A}_r(U)' = \{Y \in \mathcal{B}(\mathcal{K});\ [I_{\mathcal{H}} \otimes Y, U^*(X \otimes I_{\mathcal{K}})U] = 0 \quad \forall X \in \mathcal{B}(\mathcal{H})\}.$$

On remarque alors, en regardant le cas précédent où $\mathcal{A}(U)$ est commutative, que $\mathcal{A}_r(V) = \mathcal{A}_r(U) = \mathcal{A}(U)$. On a donc réussit à caractériser la propriété classique de l'environnement chez V à l'aide de l'algèbre $\mathcal{A}_r(V)$. Plus formellement, je prouve la chose suivante :

Théorème III.3 (Théorème 3.1)**.** *Soit* U *un opérateur unitaire sur* $\mathcal{H} \otimes \mathcal{K}$. *Supposons que* $\mathcal{H}$ *soit de dimension finie et que* $\mathcal{A}_r(U)$ *soit une algèbre de von Neumann de type I. Alors :*

1. *Il existe un opérateur unitaire* V *sur* $\mathcal{H} \otimes \mathcal{K}$ *tel que pour tout* $X \in \mathcal{B}(\mathcal{H})$,

$$V^*(X \otimes I_{\mathcal{K}})V = U^*(X \otimes I_{\mathcal{K}})U,$$

et tel que

$$\mathcal{A}(V) = \mathcal{A}_r(V) = \mathcal{A}_r(U).$$

2. *Si* V_1, V_2 *conviennent, alors* $V_1 V_2^* \in I_{\mathcal{H}} \otimes \mathcal{A}_r(U)$.

De ce théorème, on peut en déduire le résultat voulu.

Corollaire III.1. *Supposons que* $\mathcal{H}$ *et* $\mathcal{K}$ *soient de dimensions finies, de dimensions respectives* N *et* d. *Soit* U *un opérateur unitaire sur* $\mathcal{H} \otimes \mathcal{K}$. *Alors* $\mathcal{A}_r(U)$ *est commutative si et seulement si il existe deux bases orthonormées* (φ_i) *et* (ψ_i) *sur* $\mathcal{K}$ *et des opérateurs unitaires* $U_1, ..., U_d$ *sur* $\mathcal{H}$ *tels que :*

$$U = \sum_{i=1}^{d} U_i \otimes |\varphi_i\rangle\langle\psi_i|.$$

Nous avons ainsi pu complètement caractériser à l'aide de l'algèbre de l'action à droite de l'environnement les situations où l'on aimerait dire que l'environnement à une action classique sur le système.

III. 3 Bruits quantiques, intégrales stochastiques quantiques

Nous présentons maintenant la théorie permettant de dilater tout semi-groupe de Markov quantique $\mathcal{P}$, appelée *Calcul Stochastique Quantique*. Comme nous l'avons déjà dit, le but de cette théorie est d'intégrer l'environnement dans la description de la dynamique markovienne d'un système quantique ouvert. L'espace des états qui modélise celui-ci est l'*espace de Fock probabiliste* $\Phi(\mathbb{C}^d)$, c'est à dire l'espace de Fock symétrique sur $L^2(\mathbb{R}^+, \mathbb{C}^d)$:

$$\Phi(\mathbb{C}^d) = \Gamma_s(L^2(\mathbb{R}^+, \mathbb{C}^d)).$$

Rappelons maintenant les notations concernant la théorie d'intégration développée sur cette espace.
Notons $\Lambda = \{1, ..., d\}$ et prenons une base orthonormée $(e_i)_{i\in\Lambda}$ de $\mathbb{C}^d$. Sur $\Phi(\mathbb{C}^d)$, on note respectivement $A(f)$ et $A^*(f)$ les opérateurs d'annihilation et de création, pour $f \in L^2(\mathbb{R}^+, \mathbb{C}^d)$; on note de même $d\Gamma(H)$ l'opérateur différentiel de seconde quantification, où $H \in \mathcal{B}(\mathbb{C}^d)$ est un opérateur normal. Les *bruits quantiques* $a^i_j(t)$, $i, j \in \Lambda \cup \{0\}$, sont alors définis comme suit :

$$a^0_i(t) = A^*(\mathbb{1}_{[0,t]}|e_i\rangle),$$
$$a^i_0(t) = A(\mathbb{1}_{[0,t]}\langle e_i|),$$
$$a^i_j(t) = d\Gamma(\mathbb{1}_{[0,t]}|e_j\rangle\langle e_i|),$$

où $\mathbb{1}_{[0,t]}$ est la fonction indicatrice de l'intervalle $[0, t]$. Rappelons que $\Phi(\mathbb{C})$ est isométrique aux espaces L^2 canoniques du mouvement brownien et du processus de Poisson compensé de paramètre $\lambda > 0$, grâce à la *propriété de représentation chaotique* de ces espaces. Via cette isométrie :

- l'opérateur de multiplication M_{B_t} par la i-ième coordonnée du mouvement brownien s'identifie à l'opérateur $a^0_i(t) + a^i_0(t)$;
- l'opérateur de multiplication $M_{N^i_t}$ par la i-ième coordonnée du processus de Poisson compensé d'intensité λ s'identifie à l'opérateur $a^0_i(t) + a^i_0(t) + a^i_i(t)$.

Il est remarquable que ces deux processus sont les deux seuls processus à accroissements stationnaires connus possédant la propriété de représentation canonique. Il est donc naturelle que ce soit les deux seuls bruits possibles apparaissant dans l'équation de Langevin quantique.

Sur l'espace $\mathcal{B}(\mathcal{H}) \otimes \Phi(\mathbb{C}^d)$, l'équation suivante est appelée *équation stochastique quantique* :

$$dU_t = \sum_{i,j \in \Lambda \cup \{0\}} L^i_j U_t da^i_j(t), \tag{III.8}$$

où les L^i_j sont des opérateurs sur $\mathcal{H}$. Le cas particulier où la solution est une famille d'unitaire correspond à l'équation de Langevin quantique qui nous intéresse. Cela implique des contraintes particulières sur les coefficients que nous ne détaillerons pas ici.

Exemples III.2. Prenons le cas simple où $d = 1$. L'équation de Langevin quantique s'écrit alors :

$$dU_t = \left(iH - \frac{1}{2}L^*L\right) U_t\, dt + L\, U_t\, da^0_1(t) - L^* S\, U_t\, da^1_0(t) + (S - I)\, U_t\, da^1_1(t)\,, \tag{III.9}$$

où H est un opérateur autoadjoint et S est un opérateur unitaire. Les deux cas qui nous intéressent sont les suivant :

- si $S = I_{\mathcal{H}}$ et $L = -L^*$, alors l'équation prend la forme :

$$dU_t = -(iH + \frac{1}{2}L^2)U_t dt + LU_t dB_t, \tag{III.10}$$

  où $B_t := a^0_1(t) + a^1_0(t)$ est isomorphe à l'opérateur de multiplication par un mouvement brownien réel ;

- si $L = \rho(S - I_{\mathcal{H}})$ $(\rho > 0)$, alors l'équation prend la forme

$$dU_t = -(iH + \frac{1}{2}\rho^2(2I_{\mathcal{H}} - S - S^*))U_t dt + \rho(S - I_{\mathcal{H}})U_t dN_t, \tag{III.11}$$

  où $N_t := a^0_1(t) + a^1_0(t) + \frac{1}{\rho} da^1_1(t)$ est isomorphe à l'opérateur de multiplication par un processus de Poisson compensé d'intensité ρ^2.

On voit dans l'exemple précédent comment dans certains cas les coefficients peuvent se combiner de tel sorte que les bruits quantiques forment un bruit classique. Un point crucial dans notre étude avec Stéphane Attal est que l'équation (III.8) dépend du choix d'une base orthonormée de $\mathbb{C}^d$. Plus précisément, les coefficients dans l'équation en dépendent. Si on change de base, les coefficients changent aussi ; cependant, les conditions pour que la solution soit unitaire restent vérifier. Nous avons appelé une telle opération un *changement de bruit*.

III. 4 Présentation de l'article : Parties classiques et quantiques de l'environnement dans l'équation de Langevin quantique

Il était plus ou moins connu que les seuls bruits admissibles dans l'équation de Langevin classique sont les processus browniens et les processus de Poisson compensés. Cependant, nous n'avons pu trouver aucune preuve rigoureuse de ce fait dans la littérature. Notre première motivation fut donc d'établir rigoureusement ce résultat. Pour ce fait, nous utilisons encore une algèbre de von Neumann sur l'environnement.

Dans la suite, $(U_t)_{t\geq 0}$ désignera toujours l'unique solution unitaire de l'équation (III.8).

Définition III.4 (Algèbre du bruit en temps continu)**.** *Nous définissons l'*Algèbre du Bruit *au temps $t > 0$ comme la plus petite algèbre de von Neumann $\mathcal{A}_t(U)$ telle que*

$$U_s \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{A}_t(U) \text{ pour tout } s \leq t.$$

De nouveau, nous pouvons grâce à cette algèbre dire que l'environnement est *classique* si $\mathcal{A}_t(U)$ est commutative pour tout $t > 0$. Dans l'article, nous prouvons que c'est le cas si et seulement si il est possible de faire un changement de bruit tel que seuls des bruits brownien ou poissoniens apparaissent dans l'équation. De plus, nous avons une expression explicite des coefficients qui permettent d'obtenir de tels bruits.

L'algèbre du bruit nous permet une nouvelle fois, dans le cas général, d'identifier les parties classiques de l'environnement. En outre, nous sommes encore en capacité de prouver facilement une décomposition entre une partie classique et une partie quantique.

Malheureusement, la décomposition que nous obtenons n'est pas explicite. Pour le moment, nous n'avons pas d'algorithme nous permettant, étant donné une équation avec ses coefficients, de retrouver la partie commutative et la partie quantique. Ce point est illustré par un exemple à la fin du chapitre 2.

III. 5 Conclusion et perspectives pour la partie A

Nous avons vu que dans l'étude des bruits quantiques intervenant aussi bien en temps discret qu'en temps continu, il est possible de décrire la structure de ces bruits à l'aide d'une algèbre de von Neumann. Cette algèbre encode, dans le cas commutatif, toute l'information sur la nature stochastique du processus.

Nous pouvons ainsi identifier la nature des bruits classiques, ainsi que dans le cas général les parties classiques de l'environnement.

Il reste cependant plusieurs questions en suspens. Certaines sont détaillées dans le chapitre 2, sur lesquelles je ne reviendrai pas ici. Dans le futur, une problématique que je juge importante à laquelle je voudrais m'atteler serait d'identifier de manière plus systématiques les bruits apparaissant dans l'équation de Langevin quantique. Nous avons ainsi résolu le cas commutatif, mais serait-il possible d'identifier d'autres processus quantiques connus - citons par exemples les processus q-browniens [BKS97] [BS91].

IV Présentation de la partie II : Dynamiques sous-classiques et marches quantiques ouvertes

Dans la partie précédente, nous avons vu comment l'environnement pouvait générer des bruits classiques. Dans cette partie, la dynamique classique émerge à l'intérieur du système. Soit $(\Omega, \mathcal{F}, \mathbb{P})$ un espace de probabilité. On note $\mathcal{H} = L^2(\Omega, \mathcal{F}, \mathbb{P})$ et

$$\mathcal{A} = \left\{M_f;\ f \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})\right\}.$$

L'espace de probabilité que l'on considère ici est $\mathcal{B}(\mathcal{H})$ muni de l'état donné par la matrice densité $|\mathbb{1}\rangle\langle\mathbb{1}|$, où $\mathbb{1}$ est la fonction constante égale à 1.

Définition IV.1. *Soit $(\mathcal{P}_t)_{t\geq 0}$ un semi-groupe de Markov quantique sur $\mathcal{B}(\mathcal{H})$. On dit que $\mathcal{P}$ est $\mathcal{A}$-sous-classique si $\mathcal{P}$ laisse $\mathcal{A}$ invariant.*

La conséquence immédiate de cette définition est qu'il existe alors un semi-groupe de Markov classique $(P_t)_{t\geq 0}$ sur $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ tel que pour tout $f \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ et pour tout temps $t \geq 0$,

$$\mathcal{P}_t(M_f) = M_{P_t f}.$$

La dynamique d'une observable appartenant à l'algèbre $\mathcal{A}$ est ainsi classique. Dans le chapitre 3, nous étudierons de tels semi-groupes. Nous proposerons une méthode pour construire des exemples en partant d'un semi-groupe de Markov classique. Nous appliquons ensuite cette méthode aux semi-groupes de Lévy.

Dans le cas particulier où Ω est discret, on peut considérer chaque point de Ω comme le sommet d'un graphe. On note alors $V = \Omega$. Il existe alors une espérance conditionnelle naturelle de $\mathcal{B}(\mathcal{H})$ dans $\mathcal{A}$, au sens des algèbres de von Neumann. Détaillons ce point.

Notons $(|x\rangle)_{x\in V}$ la base orthonormée canonique de $\mathcal{H} = l^2(V)$ et définissons l'application $E_{\mathcal{A}}$ sur $\mathcal{B}(\mathcal{H})$ par :

$$X \in \mathcal{B}(\mathcal{H}) \mapsto E_{\mathcal{A}}(X) = \sum_{x\in V} \langle x|X|x\rangle |x\rangle\langle x|.$$

Soit μ une mesure de probabilité sur V. On peut facilement définir un état sur $\mathcal{B}(\mathcal{H})$ qui étend μ, dont la matrice densité ω_μ est donnée par

$$\omega_\mu = \sum_{x\in V} \mu(x)|x\rangle\langle x|.$$

On vérifie immédiatement que $\mathrm{Tr}[\omega_\mu E_{\mathcal{A}}(\cdot)] = \mathrm{Tr}[\omega_\mu \cdot]$, c'est à dire que $E_{\mathcal{A}}$ est compatible avec tous ces états. De plus, on peut facilement caractériser la propriété d'être sous-classique par rapport à $E_{\mathcal{A}}$. En effet, une application complètement positive $\mathcal{L}$ est $\mathcal{A}$ sous-classique si et seulement si

$$E_{\mathcal{A}} \circ \mathcal{L} \circ E_{\mathcal{N}} = \mathcal{L} \circ E_{\mathcal{N}}.$$

Un cas particulier est quand $\mathcal{L}$ obéit à la condition plus forte

$$E_{\mathcal{N}} \circ \mathcal{L} = \mathcal{L} \circ E_{\mathcal{N}} = \mathcal{L}.$$

Dans ce cas, on voit que non seulement $\mathcal{L}$ préserve $\mathcal{A}$ - les opérateurs diagonaux dans la base des $(|x\rangle)_{x\in V}$- mais aussi annule tous les opérateurs hors diagonaux. Dans un sens, cette évolution est *purement classique*, vu que complètement diriger par la dynamique sur $\mathcal{A}$.

Je voudrais maintenant expliquer le lien qui existe entre les dynamiques quantiques précédentes et les *marches quantiques ouvertes* (OQW), définies par Attal et ses collaborateurs dans [APSS12].

Considérons un autre espace de Hilbert $\mathcal{K}$ et un canal quantique $\mathfrak{M}$ sur $\mathcal{L}_1(\mathcal{K} \otimes \mathcal{H})$, c'est à dire une application complètement positive qui préserve la trace. Une définition équivalente des OQW est alors la suivante. $\mathfrak{M}$ *est une OQW ssi* :

$$(I \otimes E_{\mathcal{A}}) \circ \mathfrak{M} = \mathfrak{M} \circ (I \otimes E_{\mathcal{A}}) = \mathfrak{M}.$$

On voit la similitude avec les évolutions purement classiques mentionnées plus haut. Mais dans ce cas, la dynamique est régie par les opérateurs diagonaux par blocs.

Nous pouvons généraliser la définition précédente des OQW de la manière suivante. Sur un espace de Hilbert $\mathcal{H}$ quelconque, soit $\mathcal{M}$ une sous-algèbre de von Neumann de

$\mathcal{B}(\mathcal{H})$ telle qu'il existe une espérance conditionnelle normale $E_{\mathcal{M}}$ sur $\mathcal{B}(\mathcal{H})$ et d'image $\mathcal{M}$. Nécessairement, d'après un résultat de Tomiyama [Tom59], l'algèbre $\mathcal{M}$ est atomique : il existe une famille dénombrable de projections orthogonales $(P_x)_{x\in V}$ mutuellement orthogonales et telle que $\sum_{x\in V} P_x = I_{\mathcal{H}}$ et

$$\mathcal{M} = \sum_{x\in V} P_x \mathcal{M} P_x,$$

où les algèbres $P_x\mathcal{M}P_x$ sont des facteurs (V n'est pas forcément le même que ci-dessus, nous gardons la même notation pour insister sur l'analogie). Nous pouvons alors appeler *marche quantique ouverte généralisée* tout canal quantique $\mathfrak{M}$ sur $\mathcal{L}_1(\mathcal{H})$ qui vérifie

$$E_{\mathcal{M}} \circ \mathfrak{M} = \mathfrak{M} \circ E_{\mathcal{M}} = \mathfrak{M}.$$

Dans le chapitre 4, nous appellerons OQW de tels canaux quantiques, avec l'hypothèse supplémentaire que $P_x\mathcal{M}P_x$ est *spatialement isomorphique* à $\mathcal{B}(\mathfrak{h}_x)$ pour un certain espace de Hilbert $\mathfrak{h}_x$, ce qui implique que $\mathcal{H} = \bigoplus_{x\in V} \mathfrak{h}_x$. Pour de tel canaux quantiques, nous définissons un problème de Dirichlet associé au domaine V et à un sous-domaine $D \subset V$, que nous résolvons à l'aide d'une approche variationnelle.

V Présentation de la partie III : Décohérence Induite par l'Environnement

Cette partie, constituée d'un seul chapitre, se consacre à l'étude de la Décohérence Induite par l'Environnement (EID) pour des semi-groupes de Markov quantiques. La définition mathématique rigoureuse de l'EID dans le cas de système markovien fut introduite par Blanchard et Olkiewicz dans [BO03]. Soit $(\mathcal{P}_t)_{t\geq 0}$ un semi-groupe de Markov quantique $*$-faiblement continu et agissant sur une algèbre de von Neumann $\mathcal{M}$.

Definition V.1. *On dit qu'il y a* décohérence induite par l'environnement *si il existe une décomposition de $\mathcal{M}$,*

$$\mathcal{M} = \mathcal{M}_0 \oplus \mathcal{M}_1, \tag{V.1}$$

telle que :

- *$\mathcal{M}_0$ et $\mathcal{M}_1$ sont tous les deux $\mathcal{P}$-invariants.*
- *$\mathcal{M}_0$ est l'algèbre maximale sur laquelle $\mathcal{P}$ agit comme un $*$-automorphisme.*
- *$\mathcal{M}_1$ est un sous-espace non-vide, $*$-faiblement fermé et $*$-invariant de $\mathcal{B}(\mathcal{H})$.*

- $\mathcal{P}_t(X)$ *converge* $*$*-faiblement vers* 0 *si* $X \in \mathcal{M}_1$.

On appelle $\mathcal{M}_0$ *l'*algèbre des observables effectives*.*

On peut donner l'interprétation suivante à cette définition. $\mathcal{M}_1$ représente la partie du système qui n'est pas accessible par l'expérimentation : en effet, si la décohérence est assez rapide, toute mesure d'une observable $X \in \mathcal{M}_1$ donnera la valeur 0. Ainsi, en temps long, le système se comporte comme un système fermé modélisé par $\mathcal{M}_0$.

On voit tout de suite apparaître deux questions fondamentales. Tout d'abord, quand a-t-on EID ? Ensuite, à quelle vitesse ce processus se produit ? Dans le chapitre 5 nous prouvons la chose suivante : *l'EID a toujours lieu si* $\mathcal{M}$ *est une algèbre de von Neumann finie, et si* $\mathcal{P}$ *possède un état invariant normal et fidèle.*

Dans le même chapitre, nous nous intéressons ensuite à la vitesse de décohérence, ce qui constitue pour moi la partie la plus intéressante de ce chapitre. Pour expliquer mes motivations, on va regarder un exemple particulier pour lequel il est possible de tout expliciter. Considérons un système quantique bipartie, $\mathbb{C}^N \otimes \mathbb{C}^d$, qui évolue de la manière suivante :

- $\mathcal{H}$ est un système isolé qui évolue de manière unitaire selon le groupe unitaire $(U_t = e^{-itH})_{t\in\mathbb{R}}$, où $H \in \mathcal{B}(\mathcal{H})$ est un opérateur autoadjoint. Les états invariants pour cette évolution sont alors donnés par les matrices densités η qui commutent avec H.

- $\mathcal{K}$ est un système ouvert qui subit une dissipation, modélisé par le canal quantique dépolarisant $(\mathcal{P}_t^\tau)_{t\geq 0}$, dont le générateur est donné par :

  $$\mathcal{L}^\tau(Y) = \mathrm{Tr}[\tau Y] I_{\mathcal{K}} - Y \qquad Y \in \mathcal{B}(\mathcal{K}), \tag{V.2}$$

  où τ est une matrice densité sur $\mathcal{K}$ fixée. Il est clair que si τ est inversible, alors il est l'unique état invariant de $\mathcal{P}^\tau$ et que ce dernier est *primitif*, c'est à dire $\lim_{t\to+\infty} \mathcal{P}_t(Y) = \mathrm{Tr}[\tau Y] I_{\mathcal{K}}$ pour tout $Y \in \mathcal{B}(\mathcal{K})$.

  Dans la suite, nous supposons donc que τ *est inversible*, de telle sorte que l'état qu'il définit sur $\mathcal{K}$ est fidèle.

Le système entier évolue donc suivant le semi-groupe de Markov quantique $\mathcal{P}$ tel que pour tout $X \in \mathcal{B}(\mathcal{H})$ et $Y \in \mathcal{B}(\mathcal{K})$,

$$\mathcal{P}_t(X \otimes Y) = \left(U_t^* X U_t\right) \otimes \mathcal{P}_t(Y) \qquad \forall t \geq 0. \tag{V.3}$$

Les états invariants de $\mathcal{P}$ sont alors donnés par des matrices densités de la forme $\eta \otimes \tau$, avec $[\eta, H] = 0$.

Intéressons-nous maintenant au phénomène de décohérence pour ce semi-groupe. On note $\mathcal{N}(\mathcal{P})$ l'algèbre de von Neumann $\mathcal{B}(\mathcal{H}) \otimes I_{\mathcal{K}}$. Elle est appelée l'algèbre *libre de décohérence* du semi-groupe $\mathcal{P}$ et vérifie :

$$\mathcal{N}(\mathcal{P}) = \left\{X \in \mathcal{B}(\mathcal{H});\ \mathcal{P}_t(X^*X) = \mathcal{P}_t(X)^*\mathcal{P}_t(X), \quad \mathcal{P}_t(XX^*) = \mathcal{P}_t(X)\mathcal{P}_t(X)^*\right\}. \qquad \text{(V.4)}$$

Il existe de plus une espérance conditionnelle naturelle de $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ sur cette algèbre donnée par

$$E_{\mathcal{N}}(X) = \mathrm{Tr}_\tau[X] \otimes I_{\mathcal{K}}, \qquad X \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K}). \qquad \text{(V.5)}$$

L'espérance conditionnelle $E_{\mathcal{N}}$ a les propriétés suivantes : elle commute avec $\mathcal{P}$ et est compatible avec tous les états invariants : pour toute matrice densité η sur $\mathcal{H}$ telle que $[\eta, H] = 0$ et pour tout $X \in \mathcal{B}(\mathcal{H})$,

$$\mathrm{Tr}\big[\,(\eta \otimes \tau)\,E_{\mathcal{N}}(X)\big] = \mathrm{Tr}\big[\,(\eta \otimes \tau)\,X\big].$$

Cette espérance conditionnelle joue un rôle central dans notre étude. En général, il est toujours possible de définir $\mathcal{N}(\mathcal{P})$ et sous l'hypothèse que $\mathcal{P}$ possède un état invariant fidèle, on peut montrer que $E_{\mathcal{N}}$ existe et vérifie la même propriété que ci-dessus [DFSU14]. Dans notre cas, on peut de plus facilement vérifier que le Lindbladien $\mathcal{L}$ de $\mathcal{P}$ est donné par la formule

$$\mathcal{L}(X) = -i[H \otimes I_{\mathcal{K}}, X] + E_{\mathcal{N}}(X) - X, \qquad X \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K}).$$

La matrice densité $\sigma_{\mathrm{Tr}} = \frac{I_{\mathcal{H}}}{N} \otimes \tau$ jouera un rôle crucial dans ce qui va suivre. Remarquons qu'elle définit un état $\mathcal{P}$-invariant fidèle sur $\mathcal{H} \otimes \mathcal{K}$.

Nous pouvons maintenant affirmer que $\mathcal{P}$ a la propriété de décohérence induite par l'environnement, avec $\mathcal{M}_0 = \mathcal{N}(\mathcal{P})$ et

$$\mathcal{M}_1 = \left\{Y \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K}); \quad \mathrm{Tr}[\sigma_{\mathrm{Tr}} XY] = 0 \qquad \forall X \in \mathcal{N}(\mathcal{P})\right\}. \qquad \text{(V.6)}$$

Dans la troisième partie du chapitre 5, nous nous intéressons à la vitesse de décohérence d'un semi-groupe de Markov. Détaillons notre approche dans le cas présent. Une première possibilité est de remarquer que la vitesse de décohérence du semi-groupe $\mathcal{P}$ est liée à la vitesse de convergence de $\mathcal{P}^\tau$ vers son état stationnaire τ. On peut donc appliquer des notions connues telles que trou spectral et constante de log-Sobolev, pour obtenir des vitesses de convergences exponentielles pour la décohérence. En particulier, les trous spectraux de $\mathcal{P}$ et $\mathcal{P}^\tau$ coïncident, étant donné que la partie unitaire de $\mathcal{P}$ ne change le spectre que d'une phase.

Le trou spectral est, dans le cas d'un semi-groupe primitif, relié à l'inégalité de Poincaré comme la constante optimale λ telle que :

$$\lambda \mathrm{Var}_\tau(X) \leq \mathcal{E}(X), \qquad X \in \mathcal{B}(\mathcal{K}),\ X = X^*, \tag{V.7}$$

où Var_τ dénote la variance selon l'état τ et $\mathcal{E}$ est la forme de Dirichlet associée au semi-groupe. Cela donne une formulation variationnelle du trou spectral qui s'avère plus pratique à vérifier que de regarder directement le spectre. De plus cette inégalité est directement équivalente à la vitesse de convergence exponentielle en norme L^2 du semi-groupe.

La première définition que j'apporte dans le chapitre 5 est une généralisation de cette inégalité adaptée à la décohérence. Pour faire court, j'introduis une nouvelle définition de la variance qui fait appel à l'espérance conditionnelle $E_\mathcal{N}$:

$$\mathrm{Var}_\mathcal{N}(X) = \|X - E_\mathcal{N}(X)\|^2_{\sigma_{\mathrm{Tr}}}, \qquad X \in \mathcal{B}(\mathcal{K}),$$

et qui fait explicitement appel à la matrice densité σ_{Tr} via la norme L^2 considérée. En échangeant les deux variances dans l'Inégalité (V.7), on retrouve le trou spectral. Cette approche a donc le mérite d'être intrinsèque et de ne pas faire appel à la structure particulière du semi-groupe et de l'espace sur lequel il est défini ; elle se généralise d'ailleurs très bien au cas d'une algèbre de von Neumann finie.

Le principal désavantage de la méthode ci-dessus est qu'elle ne tient pas en compte l'espace $\mathcal{H}$ et de l'intrication que celui-ci peut avoir avec $\mathcal{K}$. En effet, quelque soit $\mathcal{H}$ et la dynamique unitaire $(U_t)_{t\geq 0}$, le trou spectral de $\mathcal{P}$ reste le même et ne dépend que de $\mathcal{P}^\tau$. Il en serait évidemment de même de toute quantité ne dépendant que de $\mathcal{P}^\tau$.
Dans un second temps, je définis donc une notion d'entropie, que j'appelle l'*entropie libre de décohérence*, qui comme pour $\mathrm{Var}_\mathcal{N}$ ne fait intervenir que l'espérance conditionnelle $E_\mathcal{N}$ et l'état σ_{Tr}, deux objets intrinsèques au semi-groupe. Cette entropie ne s'annule que sur $\mathcal{N}(\mathcal{P})$ et est donc un bon candidat pour mesurer la distance à cette algèbre. Je lui associe donc de manière naturelle une inégalité de log-Sobolev, qui implique une décroissance exponentielle du semi-groupe.
Finalement, je montre que la constante de log-Sobolev associée est plus petite que la même constante log Sobolev correspondante à $\mathcal{P}_\tau$; elle est donc plus petite que le trou spectral. *La différence entre ces deux constantes est une manifestation de l'intrication entre les deux systèmes.* Elle représente en quelque sorte la quantité d'information qui est stockée par intrication dans le système et qui est relâché dans l'environnement lors du processus

de décohérence.

La constante de log-Sobolev de $\mathcal{P}^\tau$ a été très récemment calculée explicitement [MHFW16]. L'objectif serait donc de calculer tout du moins une majoration de la constante log-Sobolev de $\mathcal{P}$, de la comparer à celle de $\mathcal{P}^\tau$ et surtout d'étudier sa dépendance en la dimension de $\mathcal{H}$.

Part A

Classical and Quantum Parts of the Quantum Dynamics

The next two chapters present two submitted articles, both to *Annales Henry Poincaré*. They both focus on the emergence of classical noises in the environment of an open system. The first article deals with the case of a one-step unitary evolution on a bipartite system, which is a toy model for studying the continuous time case. The second article focus on this latter case.

Our main contribution is the definition of a von Neumann algebra on the environment, which is used to capture the structure of the quantum noises. Consequently, whenever this algebra is commutative, it is the one generated by a classical noise whose structure we can identify. We can thus prove rigorously a fact which was already well-known: the only classical noises that appear in the quantum Langevin Equation are Brownian and Poisson processes.

The other advantage of the algebra is that in the general case, we can prove the existence of a decomposition between a maximal classical part and a purely quantum part of the environment. This amounts to answering the question: how far is the environment from being classical. The decomposition is proved both in the one-step evolution case and the continuous time case.

- Chapter 1 has been accepted for publication in *Annales Henry Poincaré*
- Chapter 2 is a collaboration with Stéphane Attal and has been submitted to *Annales Henry Poincaré*.

Chapter 1

Article: Classical and quantum parts of the quantum dynamics: the discrete-time case

Abstract:

In the study of open quantum systems modeled by a unitary evolution of a bipartite Hilbert space, we address the question of which parts of the environment can be said to have a "classical action" on the system, in the sense of acting as a classical stochastic process. Our method relies on the definition of the *Environment Algebra*, a relevant von Neumann algebra of the environment. With this algebra we define the classical parts of the environment and prove a decomposition between a maximal classical part and a quantum part. Then we investigate what other information can be obtained via this algebra, which leads us to define a more pertinent algebra: the *Environment Action Algebra*. This second algebra is linked to the minimal Stinespring representations induced by the unitary evolution on the system. Finally in finite dimension we give a characterization of both algebras in terms of the spectrum of a certain completely positive map acting on the states of the environment.

I Introduction

In the Markovian interpretation of open quantum systems, the equation describing the evolution of a system is the sum of an Hamiltonian term and additional terms representing the noises introduced by the environment. In some cases it is possible to model the action of the environment on the system by a classical noise. This way the evolution of the system

become the average over the noise of some unitary evolutions. In a more mathematical language, the evolution of the system is given by the average evolution of a Markov process taking value in the unitary group of the system.
For instance, the master equation describing the evolution of the state of the system can be written as a Schrödinger equation perturbed by some classical noises [SBOM07]. Such a model can be fruitful as it allows to borrow powerful tools from stochastic calculus to the study of open quantum systems. This approach has already been successfully used in the study of some two-levels systems [AB08] and classical reducing [Reb09].
For such equations, Kümmerer and Maassen show in [KM87] that the evolution can be dilated as a unitary evolution on the system and its environment, such that the environment appears as a Markov process. They called such dilations *essentially commutative*. More particularly, they show that the three following assertions are equivalent when the system has a finite number of degree of freedom:

1. There exists a convolution semigroup of probability measures on the group of automorphisms of the set of linear map on the system, such that the evolution is given by the expectation of the convolution semigroup.

2. There exists an essentially commutative dilation of the Quantum Markov Semigroup.

3. The operator algebra generated by the evolution on the environment is commutative.

In this article we have a different point of view, as we start directly from a unitary evolution between the system and its environment. More particularly we focus on the case of a one-step evolution, that is, when the evolution is given by a unitary operator on a tensor product of the system and the environment (a bipartite Hilbert space).
In this situation the equivalence between points 1. and 2. has already been proven by Attal and al. Indeed in [ADP] they show which unitary operators can be written in terms of classical noises emerging from the environment and characterize such noises: the *obtuse random variables*. This provides a discrete analogue of an essentially commutative dilation. As proved in [AD10], in this case the evolution is equivalent to a random walk on the unitary group of the system. The starting point of this article is the equivalence between the two previous properties and the fact that the environment interferes with the system via a commutative algebra, which gives a more solid definition of a classical environment.
On the other hand, some evolutions are understood to be typically quantum or

non-commutative, although there is no clear definition of what this means. This is for instance the case for the Spontaneous Emission, where the one-step evolution is given by the following unitary operator on $\mathbb{C}^2 \otimes \mathbb{C}^2$:

$$U_{\text{se}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{I.1}$$

Our goal is to properly define the two situations mentioned above, that is, to make the distinction between a classical and a purely quantum environment. Thus the first goal of this article is to characterize, in terms of an operator algebraic framework, which unitary evolutions can be said to have a classical or purely quantum environment. Furthermore, we want to give a partial answer to the question: How far is the environment from being classical or purely quantum?

We answer this question by defining a relevant von Neumann algebra of the environment, that we call the *Environment Algebra*. The environment is classical, or commutative in our definition, if this algebra is commutative. Moreover with the help of this algebra we are able to define and identify the classical parts of the environment, that is the parts where the dynamics reduces to an evolution with classical environment. In the same way we can say that an environment is quantum, or far from being classical, if it has no classical part. It naturally leads us to prove a decomposition of the environment between a classical and a quantum part.

The environment algebra mentioned above allows to characterize whenever the unitary evolution can be written with classical noises. A natural question is then what other properties of the evolution can be obtained from it. However it appears that this algebra is "too big" to provide finer results. For instance in some cases a purely quantum environment can nonetheless lead to an evolution of the system driven by classical noises: a quantum environment can have a classical *action* on the system. This comes from the fact that the system does not see evolutions occurring on the environment only. Consequently two evolutions on the bipartite system can lead to the same one when restricted to the system. We define the *Environment Action Algebra* as the relevant algebra in this context. Two unitary operators that have the same action on the system will have the same Environment Action Algebra.

Under simple hypotheses, we prove that it is always possible to find a unitary operator

on the bipartite system which leads to the same evolution on the system, but whose Environment Action Algebra coincide with its Environment Algebra. In this sense, it is always possible to restrict the study to the Environment Algebra. Then, in order to illustrate its usefulness, we show a link between the Environment Action Algebra and minimal Stinespring representations.
Finally, we characterize both algebras in terms of the spectrum of a completely positive map, providing a practical way to determine both algebras.

We insist on the fact that in this article the term *classical* means that we deal, at least implicitly, with a *classical probability space*. Later it will become clear that the mathematical meaning of this word is rather *commutative*, as in *commutative algebra*. In our discussions we shall use both terms without distinction but we will prefer the latter when stating mathematical results.
Note also that the classical noises we are talking about are different in nature from the ones emerging in the context of quantum trajectories ([BG09] [BG13] [Pel10b] [Pel10a]). In the latter case, classical noises appears because of a continuous monitoring of some observable of the environment. In our case no observation is performed and consequently the emergence of classical noises has to be interpreted as a manifestation of the classical nature of the action of the environment on the system.

This article is structured as follow. In Section II we focus on the definition of the Environment Algebra and the subsequent decomposition of the environment between a classical and quantum part. We give some examples and we study the particular case where the evolution is given by an explicit Hamiltonian.
In Section III we define the Environment Action Algebra as a more relevant algebra of the environment and study its properties as mentioned above.

Notations: Throughout this paper, we make use of the following notations:

- Most of the time, we consider a unitary operator U acting on the tensored Hilbert space $\mathcal{H}\otimes\mathcal{K}$, where $\mathcal{H}$ and $\mathcal{K}$ are separable Hilbert space, modeling the system and the environment respectively.

- $\mathcal{B}(\mathcal{H})$ is the Banach space of all bounded operators on $\mathcal{H}$.

- $\mathrm{Tr}_{\mathcal{K}} : \mathcal{L}_1(\mathcal{H}\otimes\mathcal{K}) \to \mathcal{L}_1(\mathcal{H})$ (where $\mathcal{L}_1(\mathcal{H})$ is the space of trace-class operators on $\mathcal{H}$) stands for the partial trace over $\mathcal{K}$, that is, if ρ is a trace-class operator on $\mathcal{H}\otimes\mathcal{K}$

than for any $X \in \mathcal{B}(\mathcal{H})$,

$$\operatorname{Tr}\left[\operatorname{Tr}_{\mathcal{K}}[\rho] X\right] = \operatorname{Tr}\left[\rho \ (X \otimes I_{\mathcal{K}})\right].$$

Given a trace-class operator $\omega \in \mathcal{L}_1(\mathcal{K})$, the partial trace with respect to ω, denoted by Tr_ω, is the unique linear map from $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ to $\mathcal{B}(\mathcal{H})$ such that

$$\operatorname{Tr}\Big[\rho \operatorname{Tr}_\omega[X]\Big] = \operatorname{Tr}\Big[\,(\rho \otimes \omega)\, X\Big] \qquad \forall \rho \in \mathcal{L}_1(\mathcal{H}).$$

- We will also use the Dirac notations: for any elements $e, f \in \mathcal{H}$:
  - $|e\rangle$ is the linear map from $\mathbb{C}$ to $\mathcal{H}$, $\lambda \in \mathbb{C} \mapsto \lambda|e\rangle = \lambda e$. We most of the time identify the linear map $|e\rangle$ with the element e of $\mathcal{H}$.
  - $\langle e|$ is its dual, that is the linear map from $\mathcal{H}$ to $\mathbb{C}$ such that $g \in \mathcal{H} \mapsto \langle e|g = \langle e, g\rangle$;
  - and consequently $|e\rangle\langle f|$ stands for the linear operator on $\mathcal{H}$ such that $g \in \mathcal{H} \mapsto |e\rangle\langle f|g = \langle f, g\rangle e$.
- If $\mathcal{A}$ is a subset of $\mathcal{B}(\mathcal{K})$, $\mathcal{A}'$ is its commutant, that is the set

$$\mathcal{A}' = \{Y \in \mathcal{B}(\mathcal{K}), [Y, A] = 0 \text{ for all } A \in \mathcal{A}\}.$$

If $\mathcal{A}$ is a $\star$-stable set, then $\mathcal{A}'$ is a von Neumann algebra. In this case, by the Bicommutant Theorem of von Neumann [vN30], the von Neumann algebra generated by $\mathcal{A}$ is the bicommutant $\mathcal{A}''$.

II The Environment Algebra for one-step evolution

The goal of this section is to prove a decomposition of the environment between a classical part and a quantum part, in a sense we shall define later. This is done by studying the proper operator algebra of the environment, the *Environment Algebra*. In order to motivate the idea behind such a definition we first recall a result of Attal, Deschamps and Pellegrini on classical environment (see [ADP]).

Theorem II.1. *Suppose $\mathcal{K} \approx \mathbb{C}^d$ for some positive integer d. Let U be a unitary operator acting on the space $\mathcal{H} \otimes \mathcal{K}$. Then the following assertions are equivalent:*

1. *There exists d unitary operators $U_1, \ldots, U_d$ on $\mathcal{H}$ and an orthonormal basis ψ_i of $\mathcal{K}$ such that:*

$$U = \sum_{i=1}^{d} U_i \otimes |\psi_i\rangle\langle\psi_i|. \tag{II.1}$$

2. *There exists operators $A, B_1, ..., B_d \in \mathcal{B}(\mathcal{H})$ and an obtuse random variable $X = (X_1, ..., X_d)$ on $\mathbb{C}^d$ such that U can be written in some orthonormal basis of $\mathcal{K}$ as*

$$U = A \otimes I_{\mathcal{K}} + \sum_{i=1}^{d} B_i \otimes M_{X_i}; \tag{II.2}$$

where M_{X_i} is isomorphic to the multiplication operator by the coordinate random variable X_i.

Thus unitary operators such as in Equation (II.1) display strong classical behavior, as they are linked in a one to one way with some particular class of random variables: the obtuse random variables. As we shall not need them in the following, we do not wish to explain more on the matter. Instead we advise the reader to consult [ADP13] and [AP05].

Our first goal in this section is to find a characterization of Equation (II.1) that can be generalized in infinite dimension. This is the role of the Environment Algebra. It is based on the following remark: U is of the form (II.4) if and only if it belongs to a von Neumann algebra $\mathcal{B}(\mathcal{H}) \otimes \mathcal{A}$ where $\mathcal{A}$ is a *commutative* algebra on $\mathcal{K}$. Indeed, if U is of the form (II.1), then define the algebra $\mathcal{A}$ as:

$$\mathcal{A} = \left\{ \sum_{i=1}^{d} f(i) |\psi_i\rangle\langle\psi_i|, \ f \in L^\infty(\{1, ..., p\}) \right\}.$$

Then clearly $U \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{A}$. Conversely, assume that there exists a commutative von Neumann algebra $\mathcal{A}$ on $\mathcal{K}$ such that $U \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{A}$. It is well-known that this algebra takes the form:

$$\mathcal{A} = \left\{ \sum_{i=1}^{m} f(i) P_i, \ f \in L^\infty(\{1, ..., m\}) \right\}, \tag{II.3}$$

where the P_i's are uniquely defined mutually orthogonal projections that sums to the identity. As $U \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{A}$, there exist unitary operators $V_1, ..., V_m$ on $\mathcal{H}$ such that:

$$U = \sum_{i=1}^{m} V_i \otimes P_i. \tag{II.4}$$

We will show in Proposition II.2 how this equation can be generalized in any dimension. In general, we will see that the environment is commutative if and only if the Environment Algebra is commutative.

To finish this small introduction on classical environment, let us remark that there are still two additional classical behaviors connected to these unitary operators:

- When considering several steps of the evolution, one can prove that the resulting evolution is equivalent to a random walk on the unitary group of $\mathcal{H}$ (see [AD10]).

- For any density matrix ω on $\mathcal{K}$ and any observable $X \in \mathcal{B}(\mathcal{H})$, we have

$$\mathcal{L}_\omega(X) := \mathrm{Tr}_\omega\left[U^* X \otimes I_{\mathcal{K}} U\right] = \sum_{i=1}^{d} \langle \psi_i, \omega\psi_i \rangle U_i^* X U_i. \tag{II.5}$$

  The map $\mathcal{L}_\omega$ is thus a *random unitary* completely positive map (CP map), a kind of CP map which is largely study in Quantum Information Theory (see [MW09] for a review on that matter).

  It has been conjectured in [ADP] by Attal and al. that a unitary operator that gives a random unitary CP map for all density matrices of the environment must have the form (where φ_i is another orthonormal basis):

$$U = \sum_{i=1}^{d} U_i \otimes |\varphi_i\rangle\langle\psi_i|. \tag{II.6}$$

We will come back on these specific unitary operators in Section III. Indeed we will show as a corollary of Theorem III.1 that they correspond exactly to those that have a commutative *Environment Right-Action Algebra* (see Definition III.1). We will show that U has the form (II.6) if and only if this algebra is commutative.

As we already mentioned in the introduction, the Spontaneous Emission, whose evolution is described by the unitary operator U_{se} of Equation (I.1), is well-known for being a manifestation of the quantum world. We will see later that this can be interpreted in our framework as the fact that there does not exist a commutative algebra $\mathcal{A}$ on $\mathcal{K}$ such that $U_{\mathrm{se}} \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{A}$. This can be read directly on the Environment Algebra for this evolution, as we will show that it is the whole algebra $\mathcal{B}(\mathcal{K})$. Consequently both situations can be integrated in the framework of the Environment Algebra.

The general case stands in between the classical case and the quantum case, as it can occur that some subpart only of the environment displays classical behavior. The second goal of this section is to define what is a classical part of the environment. Then, in the general case, we want to be able to identify all the classical parts of the environment. Once again this will be done using the Environment Algebra.

In Subsection II. 1 below we define the *Environment Algebra* $\mathcal{A}(U)$ as a relevant subalgebra of $\mathcal{B}(\mathcal{K})$. We then use this algebra to properly define a commutative environment, and to characterize unitary operators with commutative environment. In Subsection II. 2 we define the classical parts of the environment, namely the *Commutative Subspaces of the Environment*, and we prove the existence of a maximal commutative subspace, that contains all the others. This provides us with a decomposition of the environment between a classical and a quantum part. In Subsection II. 3 we study such a decomposition on one example derived from a typical Hamiltonian.

II. 1 Definition, first characterization and first examples

Let U be a unitary operator on $\mathcal{H}\otimes\mathcal{K}$. We will need the following notation: for $f,g\in\mathcal{H}$, we define:

$$U(f,g)=\mathrm{Tr}_{|g\rangle\langle f|}[U],\ U^*(f,g)=\mathrm{Tr}_{|g\rangle\langle f|}[U^*]. \tag{II.7}$$

Those operators can be seen as pictures of U taken from $\mathcal{K}$ but with different angles.

Definition II.1. *Let U be a unitary operator on $\mathcal{H}\otimes\mathcal{K}$. We call the* Environment Algebra *the von Neumann algebra $\mathcal{A}(U)$ generated by the $U(f,g)$, that is*

$$\mathcal{A}(U)=\{U(f,g),\ U^*(f,g);\quad f,g\in\mathcal{H}\}''. \tag{II.8}$$

The point with this definition is that it fits with the following characterization.

Proposition II.1. *Let U be a unitary operator on $\mathcal{H}\otimes\mathcal{K}$. Then $\mathcal{A}(U)$ is the smallest von Neumann subalgebra of $\mathcal{B}(\mathcal{K})$ such that $U,U^*\in\mathcal{B}(\mathcal{H})\otimes\mathcal{A}(U)$, i.e. if $\mathcal{A}$ is another von Neumann algebra such that $U,U^*\in\mathcal{B}(\mathcal{H})\otimes\mathcal{A}$, then $\mathcal{A}(U)\subset\mathcal{A}$. Furthermore, its commutant is given by*

$$\mathcal{A}(U)'=\{Y\in\mathcal{B}(\mathcal{K}),\quad [I_{\mathcal{H}}\otimes Y,U]=[I_{\mathcal{H}}\otimes Y,U^*]=0\}. \tag{II.9}$$

Proof. First, if (e_i) is an orthonormal basis of $\mathcal{H}$, then U has the matrix decomposition (see [Att]):

$$U=\sum_{i,j}|e_i\rangle\langle e_j|\otimes U(e_i,e_j),$$

where the sum is strongly convergent if $\mathcal{H}$ is infinite dimensional. As the same decomposition holds for U^*, we obtain that $U,U^*\in\mathcal{B}(\mathcal{H})\otimes\mathcal{A}(U)$. Now for all $Y\in\mathcal{B}(\mathcal{K})$,

$$\begin{aligned}[I_{\mathcal{H}}\otimes Y,U]=0 &\Leftrightarrow \mathrm{Tr}_{|g\rangle\langle f|}[[I_{\mathcal{H}}\otimes Y,U]]=0 \text{ for all } f,g\in\mathcal{H}\\ &\Leftrightarrow \left[Y,\mathrm{Tr}_{|g\rangle\langle f|}[U]\right]=0 \text{ for all } f,g\in\mathcal{H}\\ &\Leftrightarrow [Y,U(f,g)]=0 \text{ for all } f,g\in\mathcal{H}.\end{aligned}$$

Similarly $[I_{\mathcal{H}}\otimes Y,U^*]=0$ if and only if $[Y,U^*(f,g)]=0$ for all $f,g\in\mathcal{H}$ which proves Equality (II.10).

Now suppose that $U,U^*\in\mathcal{B}(\mathcal{H})\otimes\mathcal{A}$ for some von Neumann subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{K})$. Then for all $Y\in\mathcal{A}'$, $[I_{\mathcal{H}}\otimes Y,U]=[I_{\mathcal{H}}\otimes Y,U^*]=0$ so that by the previous equality $Y\in\mathcal{A}(U)'$. Consequently $\mathcal{A}'\subset\mathcal{A}(U)'$ and then $\mathcal{A}(U)\subset\mathcal{A}$ by the Bicommutant Theorem. □

We now show how those definitions apply to the two examples mentioned in the introduction: the general situation of a commutative algebra and the particular case of the spontaneous emission.

The case of a Commutative Environment: In this paragraph we characterize the unitary operators with commutative Environment Algebra. The expression in the general case is of the same form as Equation (II.4), where the sum has to be replaced by an integral over a spectral measure. Note that in general, if a von Neumann algebra $\mathcal{A}$ on $\mathcal{K}$ is commutative, then there exist a measured space $(\Omega, \mathcal{F})$ and a spectral measure ξ on $(\Omega, \mathcal{F})$ with values in the orthogonal projections of $\mathcal{K}$ such that:

$$\mathcal{A} = \left\{ \int_\Omega f(\omega)\xi(d\omega), \quad f \in L^\infty(\Omega, \mathcal{F}) \right\}. \tag{II.10}$$

As $U \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{A}(U)$, the goal is to understand the nature of this last algebra. The next proposition shows that it consists of operators of the form (II.10) where the function f has to be replaced by a measurable family of bounded operators on $\mathcal{H}$. As $\mathcal{B}(\mathcal{H})$ can be infinite-dimensional, we need to specify a form of measurability. We say that a family of operators $(A(\omega))_{\omega\in\Omega}$ is $\mathcal{F}$-measurable if for all $f, g \in \mathcal{H}$, $\omega \mapsto \langle f, A(\omega)g \rangle$ is $\mathcal{F}$-measurable (this is usually called the weak-measurability).

Proposition II.2. *Let U be a unitary operator on $\mathcal{H} \otimes \mathcal{K}$. Then $\mathcal{A}(U)$ is commutative if and only if there exist:*

- *a measured space $(\Omega, \mathcal{F})$,*
- *a spectral measure ξ on $(\Omega, \mathcal{F})$ with values in the orthogonal projections of $\mathcal{K}$,*
- *a $\mathcal{F}$-measurable family of ξ-almost surely unitary operators $(V(\omega))_{\omega\in\Omega}$ on $\mathcal{H}$ such that:*

$$U = \int_\Omega V(\omega) \otimes \xi(d\omega). \tag{II.11}$$

If $\mathcal{K}$ is finite dimensional, then Equation (II.11) takes the simpler form of Equation (II.4).

Proof. If $\mathcal{A}(U)$ is commutative then it is of the same form as in Equation (II.10).
The first step of the proof is to construct the kind of operators that appear in Equation (II.11). Let $B(\Omega)$ denote the set of ξ-almost bounded families of bounded operators on

$\mathcal{H}$ indexed by Ω. Thus $A \in B(\Omega)$ is a random variable on $(\Omega, \mathcal{F})$ with value in $\mathcal{B}(\mathcal{H})$ and such that there exists a constant $C > 0$ with $\|A\| < C$, ξ-almost surely.

We want to integrate with respect to ξ the elements of $B(\Omega)$. For $\varphi, \psi \in \mathcal{K}$, define the complex measure $\nu_{\varphi,\psi}$ on $(\Omega, \mathcal{F})$ as

$$\nu_{\varphi,\psi} : E \in \mathcal{F} \mapsto \langle \varphi, \xi(E)\psi \rangle.$$

Then for all $A \in B(\Omega)$ and all $f, g \in \mathcal{H}$, we have

$$\begin{aligned} \left| \int_\Omega \langle f, A(\omega) g\rangle \nu_{\varphi,\psi}(d\omega) \right| &\leq \int_\Omega |\langle f, A(\omega) g\rangle| \, |\langle \varphi, \xi(d\omega)\psi\rangle| \\ &\leq \|A\| \, \|f\| \, \|g\| \, \|\varphi\| \, \|\psi\| \, . \end{aligned}$$

By Riesz representation Theorem [RS80] this defines a bounded operator on $\mathcal{H} \otimes \mathcal{K}$ that we write $\int_\Omega A \otimes d\xi$ and such that

$$\langle f \otimes \varphi, \left(\int_\Omega A(\omega) \otimes d\xi \right) g \otimes \psi \rangle = \int_\Omega \langle f, A(\omega) g \rangle \nu_{\varphi,\psi}(d\omega). \tag{II.12}$$

We call $\mathcal{B}(\xi)$ the set of operators defined by Equation (II.12). Note that elements of $\mathcal{B}(\xi)$ can also be defined as weak limits of operators of the form

$$\sum_{j \in J} A_j \otimes \xi(E_j), \tag{II.13}$$

where J is a finite set, $(A_j)_{j\in J}$ is a family in $\mathcal{B}(\mathcal{H})$ and $(E_j)_{j\in J}$ is a measurable partition of $(\Omega, \mathcal{F})$. From this remark and Equation (II.12), it is an easy exercise to check that $\mathcal{B}(\xi)$ is an algebra and that it is closed for the weak topology, so that it is a von Neumann algebra on $\mathcal{H} \otimes \mathcal{K}$. Besides, if $\int_\Omega A \otimes d\xi = \int_\Omega B \otimes d\xi$ for $A, B \in \mathcal{B}(\xi)$, then $A = B$, ξ-almost surely.

Note that if there exists $V \in B(\Omega)$ such that $U = \int_\Omega V \otimes d\xi$, then, from $UU^* = U^*U = I_{\mathcal{H}\otimes\mathcal{K}}$, we get $VV^* = V^*V = I_{\mathcal{H}}$, ξ-almost surely, which shows that the $V(\omega)$ are ξ-almost surely unitary operators on $\mathcal{H}$. Consequently, as U belongs to $\mathcal{B}(\mathcal{H}) \otimes \mathcal{A}(U)$, in order to complete the proof we only have to show that

$$\mathcal{B}(\xi) = \mathcal{B}(\mathcal{H}) \otimes \mathcal{A}(U).$$

We first prove that $\mathcal{B}(\xi) \subset \mathcal{B}(\mathcal{H}) \otimes \mathcal{A}(U)$. By the Bicommutant Theorem, this is equivalent to $I_{\mathcal{H}} \otimes \mathcal{A}(U)' \subset \mathcal{B}(\xi)'$. Let $Y \in \mathcal{A}(U)'$. It is enough to prove that $I_{\mathcal{H}} \otimes Y$ commutes with elements of the form (II.13), which is straightforward.

To prove the other inclusion, take $X \in \mathcal{B}(\mathcal{H})$ and $Y \in \mathcal{A}(U)$. Then there exists a bounded measurable function f on $(\Omega, \mathcal{F})$ such that $Y = \int_\Omega f d\xi$. Then

$$X \otimes Y = \int_\Omega (f(\omega) X) \otimes \xi(d\omega) \in \mathcal{B}(\xi).$$

This concludes the proof as the von Neumann algebra $\mathcal{B}(\mathcal{H}) \otimes \mathcal{A}(U)$ is the weak closure of operators of the form $X \otimes Y$. □

An example of purely quantum environment: the Spontaneous Emission: Consider the unitary operator U_{se} given by Equation (I.1). We suppose that $\theta \notin 2\pi\mathbb{Z}$ so that U_{se} is not the identity operator. By common knowledge it is a purely quantum evolution, so the corresponding environment algebra should not be commutative. More particularly, as $\mathcal{K} = \mathbb{C}^2$, it should be the whole algebra.

In order to check this, take an operator $Y \in \mathcal{A}(U_{\text{se}})'$. Let (e_0, e_1) be the canonical basis of $\mathbb{C}^2$ and identify Y with the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a, b, c, d \in \mathbb{C}$. The operator $I_{\mathcal{H}} \otimes Y$ is identified with the 4×4-matrix $\begin{pmatrix} aI_{\mathcal{H}} & bI_{\mathcal{H}} \\ cI_{\mathcal{H}} & dI_{\mathcal{H}} \end{pmatrix}$. Computing $[I_{\mathcal{H}} \otimes Y, U_{\text{se}}]$ leads to:

$$[I_{\mathcal{H}} \otimes Y, U_{\text{se}}] = \begin{pmatrix} 0 & b\sin\theta & b(\cos\theta - 1) & 0 \\ c\sin\theta & 0 & (d-a)\sin\theta & b(1-\cos\theta) \\ c(1-\cos\theta) & (d-a)\sin\theta & 0 & -b\sin\theta \\ 0 & c(\cos\theta - 1) & -c\sin\theta & 0 \end{pmatrix}.$$

Hence the condition $[I_{\mathcal{H}} \otimes Y, U_{\text{se}}] = 0$ implies that $b = c = 0$ and $a = d$, so that $Y = aI_{\mathcal{K}}$. Thus $\mathcal{A}(U_{\text{se}})' = \mathbb{C}I_{\mathcal{K}}$ and consequently

$$\mathcal{A}(U_{\text{se}}) = \mathcal{B}(\mathbb{C}^2).$$

II. 2 Classical and Quantum parts of the Environment

We now focus on our main topic. We have already given some examples of the algebra $\mathcal{A}(U)$ and emphasized on the particular case where it is commutative. In this subsection we state our main result for a general one-step unitary evolution: the environment can be splited into the sum of a classical and a quantum part. First using the algebra $\mathcal{A}(U)$ it is now possible to properly define what we call a classical part, or commutative part, of the environment. For instance, consider the following unitary operator U_{ex} on $\mathbb{C}^2 \otimes \mathbb{C}^4$ $(\alpha, \beta \in \mathbb{R},\ \alpha \neq \beta)$, written in the canonical orthonormal basis (e_1, e_2, e_3, e_4) of $\mathcal{K} = \mathbb{C}^4$:

$$U_{\text{ex}} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos\beta & -\sin\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & \sin\beta & \cos\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{II.14}$$

Clearly the environment is the sum of two parts $\mathcal{K}_c = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ and $\mathcal{K}_q = \mathbb{C}e_3 \oplus \mathbb{C}e_4$, such that $\mathcal{H} \otimes \mathcal{K}_{c,q}$ are stable by U and:

- The restriction U_c of U on $\mathcal{H} \otimes \mathcal{K}_c$ has a commutative environment. Indeed:

$$\begin{aligned} U_c &= \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & \cos\beta & -\sin\beta \\ 0 & 0 & \sin\beta & \cos\beta \end{pmatrix} \\ &= \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \otimes |e_1\rangle\langle e_1| + \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \otimes |e_2\rangle\langle e_2|, \end{aligned}$$

  so that $\mathcal{A}(U_c) = \mathbb{C}|e_1\rangle\langle e_1| + \mathbb{C}|e_2\rangle\langle e_2| = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix},\ a, b \in \mathbb{C} \right\}$ as $\alpha \neq \beta$.

- The restriction U_q of U on $\mathcal{H} \otimes \mathcal{K}_q$ has a quantum environment, as $U_q = U_{\text{se}}$ so that $\mathcal{A}(U_q) = \mathcal{B}(\mathcal{K}_q)$.

Consequently in this example $\mathcal{K}_c$ is a classical subspace of the environment, which is characterized by the fact that $\mathcal{H} \otimes \mathcal{K}_c$ reduces both U and U^* and that $\mathcal{A}(U_c)$ is commutative. Furthermore the other non-trivial parts of the environment that could be considered as classical, that is $\mathbb{C}e_1$ and $\mathbb{C}e_2$, are subspaces of $\mathcal{K}_c$.

This leads us to the following definition.

Definition II.2. *Let U be a unitary operator on $\mathcal{H} \otimes \mathcal{K}$ and let $\tilde{\mathcal{K}}$ be a subspace of $\mathcal{K}$. We say that $\tilde{\mathcal{K}}$ is a* Commutative Subspace of the Environment *if $\tilde{\mathcal{K}} \neq \{0\}$ and:*

- *$\mathcal{H} \otimes \tilde{\mathcal{K}}$ and $\mathcal{H} \otimes \tilde{\mathcal{K}}^\perp$ are stable by U,*

- *$\mathcal{A}(\tilde{U})$ is commutative, where $\tilde{U}$ is the restriction of U to $\mathcal{H} \otimes \tilde{\mathcal{K}}$.*

Our main result, Theorem II.2, states that there exists a maximal commutative subspace $\mathcal{K}_c$ of the environment, in the sense that all commutative subspaces of the environment are also subspaces of $\mathcal{K}_c$. The main ingredient of the proof is the following proposition.

Proposition II.3. *Let $\mathcal{A}$ be a von Neumann subalgebra of $\mathcal{B}(\mathcal{K})$. Then there exists a unique projection $P_c \in \mathcal{A}'$, possibly null, such that:*

1. *$P_c\ \mathcal{A}\ P_c$ is commutative;*
2. *if $P \in \mathcal{A}'$ is an orthogonal projection such that $P\mathcal{A}\ P$ is commutative, then $P \leq P_c$.*

This proposition is not a new result on von Neumann algebras, however we could not find it stated in this particular form in the literature. It is true for any von Neumann algebra and entirely relies on the fact that the supremum of a subclass of orthogonal projections in a von Neumann algebra still belongs to this algebra.

Proof. Define the set $\mathcal{P}_c$ of orthogonal projections in $\mathcal{A}'$ such that $P \in \mathcal{P}_c$ iff $P\mathcal{A}$ is commutative. Take $P_c = \sup \mathcal{P}_c$ the supremum over $\mathcal{P}_c$. It is again in $\mathcal{P}_c$ and it is easy to see that it fulfills the proposition. □

Our result is a direct corollary of this proposition.

Theorem II.2. *The environment Hilbert space $\mathcal{K}$ is the orthogonal direct sum of two subspaces $\mathcal{K}_c$ and $\mathcal{K}_q$, such that either $\mathcal{K}_c = \{0\}$ or:*

- *$\mathcal{K}_c$ is a commutative subspace of the environment.*
- *If $\tilde{\mathcal{K}}$ is any commutative subspace of the environment then $\tilde{\mathcal{K}}$ is a subspace of $\mathcal{K}_c$.*
- *The restriction of U to $\mathcal{H} \otimes \mathcal{K}_q$ does not have any commutative subspace.*

Proof. We take $\mathcal{K}_c = P_c\mathcal{K}$ and $\mathcal{K}_q = (I_{\mathcal{K}} - P_c)\mathcal{K} = \mathcal{K}_c^{\perp}$, where P_c is defined by applying Proposition II.3 to $\mathcal{A}(U)$.
Suppose that $P_c \neq 0$. First we check that $\mathcal{K}_c$ is a commutative subspace of the environment. By definition $P_c \in \mathcal{A}(U)'$ so that $I_{\mathcal{H}} \otimes P_c \in (\mathcal{B}(\mathcal{H}) \otimes \mathcal{A}(U))'$. As U and U^* are elements of $\mathcal{B}(\mathcal{H}) \otimes \mathcal{A}(U)$, this shows that $\mathcal{H} \otimes \mathcal{K}_c$ and $\mathcal{H} \otimes \mathcal{K}_c^{\perp}$ are stable by U. Denote by U_c the restriction of U to $\mathcal{H} \otimes \mathcal{K}_c$. Then $\mathcal{A}(U_c) = P_c\mathcal{A}(U)P_c$, which is commutative by definition of P_c.

Let $\tilde{\mathcal{K}}$ be a commutative subspace of U and denote by P the orthogonal projection on $\tilde{\mathcal{K}}$ and by $\tilde{U}$ the restriction of U to $\tilde{\mathcal{K}}$. By definition $P \in \mathcal{A}(U)'$ and $\mathcal{A}(\tilde{U}) = P\mathcal{A}(U)P$ is commutative, so $P \leq P_c$ by Proposition II.3. Consequently $\tilde{\mathcal{K}}$ is a subspace of $\mathcal{K}_c$.
Now denote by U_q the restriction of U to $\mathcal{H}\otimes\mathcal{K}_q$. By contradiction, let $\mathcal{K}'$ be a commutative subspace of U_q. By the same argument as before $\mathcal{K}'$ is also a commutative subspace of U_q. Consequently $\mathcal{K}'$ is a subspace of $\mathcal{K}_c$, so that $\tilde{\mathcal{K}} = \{0\}$, which is a contradiction. □

To conclude this subsection, let emphasize the particular situation where $\mathcal{K}$ is finite dimensional. It should not be difficult, for any von Neumann algebra $\mathcal{A}$ on $\mathcal{K}$, to construct a unitary operator U on $\mathcal{H} \otimes \mathcal{K}$ for some Hilbert space $\mathcal{H}$ such that $\mathcal{A}(U) = \mathcal{A}$, so in a sense the classification of the different environments in terms of the Environment Algebra reduces to the classification of the von Neumann algebra on $\mathcal{K}$. In this case, the structure of von Neumann algebras is well-known [KR97] and corresponds in some orthonormal basis to block-diagonal matrices, where the position and the size of the blocks characterize the algebra.
For instance, if the dimension of $\mathcal{K}$ is two, the algebra is either commutative or the whole algebra $\mathcal{B}(\mathcal{K})$. In dimension 3, the maximal commutative subspace $\mathcal{K}_c$ has dimension one or three etc,...

II. 3 Typical Hamiltonian: the Dipole Hamiltonian

In this section, $\mathcal{H}$ is a N-dimensional Hilbert space and $\mathcal{K}$ is a $(d+1)$-dimensional Hilbert space, with N, d two positive integers. If U is a unitary operator on $\mathcal{H} \otimes \mathcal{K}$ it is always possible to write it $U = \exp(-itH)$ for some selfadjoint operator H (which is not unique). Then it is interesting to study the algebra $\mathcal{A}(H)$ instead. It is also a first step towards understanding the continuous case, as the Hamiltonian describes the instantaneous evolution of the system and its environment (see [AP06]). We focus on the particular case of a dipole Hamiltonian.

Let $(e_i)_{i=0,\dots,d}$ be an orthonormal basis of $\mathcal{K}$, starting the index at 0. The vector e_0 will play a specific role in what follows. We write $\mathcal{K}' = \mathbb{C}e_0^\perp$ its orthogonal subspace. We will often identify operators on $\mathcal{K}$ with $(d+1)$-square matrices, and operators on $\mathcal{K}'$ with d-square matrices, using this basis. We also write $a_j^i = |e_j\rangle\langle e_i|$ the elementary matrices. Writing $V_j^i = \mathrm{Tr}_{\mathcal{H}}[V I_{\mathcal{H}} \otimes |e_i\rangle\langle e_j|]$, any element $V \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ can then be written as a block matrix

$$V = \sum_{i=0}^{d+1} V_j^i \otimes a_j^i = \begin{pmatrix} V_0^0 & V_0^1 & \cdots & V_0^d \\ V_1^0 & V_1^1 & \cdots & V_1^d \\ \vdots & \vdots & & \vdots \\ V_d^0 & V_d^1 & \cdots & V_d^d \end{pmatrix}.$$

In the same way elements of $\mathcal{B}(\mathcal{H})^{\oplus d}$ can be seen as columns whose components are operators on $\mathcal{H}$.

Similarly elements of the dual $(\mathcal{B}(\mathcal{H})^{\oplus d})^*$ of $\mathcal{B}(\mathcal{H})^{\oplus d}$ can be seen as rows whose components are operators on $\mathcal{H}$. If $L \in \mathcal{B}(\mathcal{H})^{\oplus d}$, we write L^Γ and L^* the following elements of $(\mathcal{B}(\mathcal{H})^{\oplus d})^*$:

$$L = \begin{pmatrix} L_1 \\ \vdots \\ L_d \end{pmatrix}, \quad L^\Gamma = \begin{pmatrix} L_1 & \cdots & L_d \end{pmatrix}, \quad L^* = \begin{pmatrix} L_1^* & \cdots & L_d^* \end{pmatrix}.$$

Elements $M = (m_{i,j})_{1\le i,j\le d}$ of $\mathcal{B}(\mathcal{K}')$ act on $\mathcal{B}(\mathcal{H})^{\oplus d}$ in the following way:

$$M : L \mapsto \begin{pmatrix} m_{1,1} I_\mathcal{H} & \cdots & m_{1,d} I_\mathcal{H} \\ \vdots & & \vdots \\ m_{d,1} I_\mathcal{H} & \cdots & m_{d,d} I_\mathcal{H} \end{pmatrix} \begin{pmatrix} L_1 \\ \vdots \\ L_d \end{pmatrix}.$$

We write this action $M \star L$, with dual action on the dual $(\mathcal{B}(\mathcal{H})^{\oplus d})^*$ of $\mathcal{B}(\mathcal{H})^{\oplus d}$:

$$L^\Gamma \star M = \begin{pmatrix} L_1 & \cdots & L_d \end{pmatrix} \begin{pmatrix} m_{1,1} I_\mathcal{H} & \cdots & m_{1,d} I_\mathcal{H} \\ \vdots & & \vdots \\ m_{d,1} I_\mathcal{H} & \cdots & m_{d,d} I_\mathcal{H} \end{pmatrix}.$$

Our result makes use of the following lemma, which is straightforward by a simple computation on block-matrices.

Lemma II.1. *Let W be a unitary operator on $\mathcal{K}'$ and write $\overrightarrow{W} = a_0^0 \oplus W$. Thus $\overrightarrow{W}$ is the unitary operator on $\mathcal{K}$ that acts as identity on $\mathbb{C}e_0$ and as W on $\mathcal{K}'$. Let $V \in \mathcal{B}(\mathcal{H}\otimes\mathcal{K})$. We write $L_0 = (V_0^1 \cdots V_0^d)$, $L^0 = (V_1^0 \cdots V_d^0)^\Gamma$ and $\mathbb{L} = (V_j^i)_{1\le i,j\le d}$, so that:*

$$V = \begin{pmatrix} V_0^0 & L_0 \\ L^0 & \mathbb{L} \end{pmatrix}.$$

Then

$$\left(I_\mathcal{H} \otimes \overrightarrow{W}\right)^* \; V \; \left(I_\mathcal{H} \otimes \overrightarrow{W}\right) = \begin{pmatrix} V_0^0 & L_0 \star W \\ W^* \star L^0 & (I_\mathcal{H} \otimes W^*)\, \mathbb{L}\, (I_\mathcal{H} \otimes W) \end{pmatrix}.$$

We focus on the typical dipole Hamiltonian usually considered in the weak coupling limit or van Hove limit [Dav74]:

$$H = H_{\mathcal{S}} \otimes I_{\mathcal{K}} + I_{\mathcal{H}} \otimes H_{\mathcal{E}} + \sum_{i=1}^{d} [V_i \otimes a_i^0 + V_i^* \otimes a_0^i],$$

where $H_{\mathcal{S}}, V_1, ..., V_d \in \mathcal{B}(\mathcal{H})$, $H_{\mathcal{E}} \in \mathcal{B}(\mathcal{K})$ and $H_{\mathcal{S}}, H_{\mathcal{E}}$ are selfadjoint operators. To simplify the notations, we write $V = \begin{pmatrix} V_1 & \cdots & V_d \end{pmatrix}^{\Gamma}$, so that we have in the orthonormal basis $(e_i)_{0 \le i \le d}$:

$$H = H_{\mathcal{S}} \otimes I_{\mathcal{K}} + I \otimes H_{\mathcal{E}} + \begin{pmatrix} 0 & V^* \\ V & 0_{\mathcal{K}'} \end{pmatrix}. \tag{II.15}$$

First remark that $\mathcal{A}(H)'$ is the commutant of the set

$$\left\{ H_{\mathcal{E}}, \ \sum_{i=1}^{d} [\langle f, V_i g \rangle \otimes a_i^0 + \langle f, V_i^* g \rangle \otimes a_0^i], \ f, g \in \mathcal{H} \right\}.$$

To make it simpler, we will suppose that $H_{\mathcal{E}} = 0$ (that is, we switch to the interaction picture of time evolution), so that

$$\mathcal{A}(H)' = \left\{ Y \in \mathcal{B}(\mathcal{K}), \ \left[I_{\mathcal{H}} \otimes Y, \begin{pmatrix} 0 & V^* \\ V & 0_{\mathcal{K}'} \end{pmatrix} \right] = 0 \right\}'. \tag{II.16}$$

Here is our result.

Theorem II.3. *Suppose $\mathcal{K} = \mathbb{C}^{d+1}$, with $d \ge 1$, and*

$$H = H_{\mathcal{S}} \otimes I_{\mathcal{K}} + \begin{pmatrix} 0 & V^* \\ V & 0_{\mathcal{K}'} \end{pmatrix},$$

for some $V = \begin{pmatrix} V_1 & \cdots & V_d \end{pmatrix}^{\Gamma} \in \mathcal{B}(\mathcal{H})^{\oplus d}$. Let m be the dimension of the subspace of $\mathcal{B}(\mathcal{H})$ generated by the V_i's. Then we are in one of the following situations:

- *either there exist $\theta \in \mathbb{R}$, $a_1, ..., a_d \in \mathbb{C}$ possibly null, such that*

$$H = H_{\mathcal{S}} \otimes I_{\mathcal{K}} + e^{-i\theta/2} V_1 \otimes \begin{pmatrix} 0 & e^{-i\theta/2}\overline{a_1} & \cdots & e^{-i\theta/2}\overline{a_d} \\ e^{i\theta/2} a_1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ e^{i\theta/2} a_d & 0 & \cdots & 0 \end{pmatrix}. \tag{II.17}$$

In this case

$$\mathcal{A}(H) = alg \left\{ \begin{pmatrix} 0 & e^{-i\theta/2}\overline{a_1} & \cdots & e^{-i\theta/2}\overline{a_d} \\ e^{i\theta/2} a_1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ e^{i\theta/2} a_d & 0 & \cdots & 0 \end{pmatrix} \right\},$$

and consequently it is commutative.

- *either $\mathcal{K}$ is the orthogonal sum of two subspaces $\mathcal{K}_1 \cong \mathbb{C}^{m+1}$ and $\mathcal{K}_2 \cong \mathbb{C}^{d-m}$, such that $\mathcal{H} \otimes \mathcal{K}_1$ and $\mathcal{H} \otimes \mathcal{K}_2$ are stable under H and $\mathcal{A}(H)$ can be decomposed as*

$$\mathcal{A}(H) = \mathcal{B}(\mathcal{K}_1) \oplus \mathbb{C} I_{\mathcal{K}_2}. \tag{II.18}$$

Before giving the proof of Theorem II.3, we introduce the following lemma which allows to reduce the problem to the minimal number of non-zero V_k and to assume their freeness.

Lemma II.2. *Suppose $\mathcal{K} = \mathbb{C}^{d+1}$, with $d \geq 1$, and*

$$H = H_{\mathcal{S}} \otimes I_{\mathcal{K}} + \begin{pmatrix} 0 & V^* \\ V & 0_{\mathcal{K}'} \end{pmatrix}, \tag{II.19}$$

for some $V = \begin{pmatrix} V_1 & \cdots & V_d \end{pmatrix}^\Gamma \in \mathcal{B}(\mathcal{H})^{\oplus d}$. Let m be the dimension of the subspace of $\mathcal{B}(\mathcal{H})$ generated by the V_i's.
Then there exists an orthonormal basis $(e'_i)_{1 \leq i \leq d}$ of $\mathcal{K}'$ such that in the new basis $(e_0, e'_1, \ldots, e'_d)$, H has the form:

$$H = H_{\mathcal{S}} \otimes I_{\mathcal{K}} + \begin{pmatrix} 0 & (V')^* \\ V' & 0_{\mathcal{K}'} \end{pmatrix}, \tag{II.20}$$

where $V' = \begin{pmatrix} V'_1 & \cdots V'_m & 0 & \cdots & 0 \end{pmatrix}^\Gamma \in \mathcal{B}(\mathcal{H})^{\oplus d}$.
Furthermore, the V'_i's are linearly independent, and $d - m$ is the maximal number of components that can be canceled.

Proof of Lemma II.2. If $m = 0$, then $V_1 = \cdots = V_d = 0$ so that the result is trivial. Suppose that $m \geq 1$.
We identify V_k with the matrix $(v_k^{i,j})_{1 \leq i,j \leq N}$ in some orthonormal basis of $\mathcal{H}$. Our goal is to find a unitary operator W on $\mathcal{K}'$ such that only the first m components of $W^* \star V$ are non-zero.
Introduce the vector $|v\rangle_{i,j} = (v_1^{i,j}, \ldots, v_d^{i,j})$ of $\mathbb{C}^d$. We will use the two following spaces:

$$\mathcal{V}_1 = \operatorname{span}\{V_1, \ldots, V_d\} \subset \mathcal{B}(\mathcal{H}),$$
$$\mathcal{V}_2 = \operatorname{span}\{|v\rangle_{i,j},\ i,j = 1, \ldots, N\} \subset \mathbb{C}^d.$$

If W is a unitary operator on $\mathcal{K}'$, we write similarly

$$V' = \begin{pmatrix} V'_1 \\ \vdots \\ V'_d \end{pmatrix} = W^* \star V,$$

where $V'_k = ((v')^{i,j}_k)_{1\leq i,j\leq N}$. If we write $|v'\rangle_{i,j} = ((v')^{i,j}_1, ..., (v')^{i,j}_d)$, then for all $i, j = 1, ..., N$

$$|v'\rangle_{i,j} = W^*|v\rangle_{i,j}.$$

Consequently, canceling the last $d-k$ components of V, for $k = 1, ..., d$, is equivalent to finding W such that for all $i, j = 1, ..., N$:

$$(v')^{i,j}_l \neq 0 \text{ for all } l = 1, ..., k,$$
$$(v')^{i,j}_l = 0 \text{ for all } l = k+1, ..., d.$$

The maximal number of components that we can cancel is thus $d - \dim \mathcal{V}_2$. Thus we have to prove that $\dim \mathcal{V}_2 = \dim \mathcal{V}_1 = m$ to finish the proof. Let A be the $N^2 \times d$-matrix whose rows are the $|v\rangle_{i,j}$. For $\lambda_1, ..., \lambda_d \in \mathbb{C}$, we have

$$\begin{pmatrix} \lambda_1 I_{\mathcal{H}} & \cdots & \lambda_d I_{\mathcal{H}} \end{pmatrix} V = 0 \quad \text{iff} \quad A \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_d \end{pmatrix} = 0.$$

Consequently, the rank of A is equal to $\dim \mathcal{V}_1 = m$. However by definition it is equal to the dimension of the subspace spanned by its rows, that is $\dim \mathcal{V}_2$. Hence the result. □

Now we can prove Theorem II.3.

Proof of Theorem II.3. If $m = 0$, the result is trivial. Suppose that $m \geq 1$.
Using Lemma II.2 we assume that only the m first V_k's are non-zero, and that they are independent. Our method in order to study $\mathcal{A}(H)$ is the following: take $Y \in \mathcal{A}(H)'$ and solve in Y the equation $[I_{\mathcal{H}} \otimes Y, H] = 0$. With our assumptions on the V_k's, it is clear that $\mathcal{K}'$ is the orthogonal direct sum of two subspaces $\mathcal{K}_1$ and $\mathcal{K}_2$, of dimension m and $d-m$ respectively, such that

$$H = H_1 \oplus 0_{\mathcal{H}\otimes\mathcal{K}_2};$$
$$\mathcal{A}(H) = \mathcal{A}(H_1) \oplus \mathbb{C} I_{\mathcal{K}_2};$$

where H_1 is the selfadjoint operator induced by H on $\mathcal{H} \otimes (\mathbb{C}e_0 \oplus \mathcal{K}_1)$. Thus we can restrict the study to H_1 only.
Take $Y \in \mathcal{A}(H_1)'$. As $\mathcal{A}(H_1)'$ is a $*$-algebra, we can assume without loss of generality that Y is selfadjoint. We identify Y with the matrix $(y_{ij})_{0\leq i,j\leq m}$ using the orthonormal basis $(e_i)_{0\leq i\leq m}$ of $\mathcal{K}_1$. Now as $y_{00} I_{\mathcal{K}} \in \mathcal{A}(H_1)'$, taking $Y - y_{00} I_{\mathcal{K}}$ we can furthermore assume that

$y_{00} = 0$. Then, solving $[I_{\mathcal{H}} \otimes Y, H_1] = 0$, we obtain in particular the following conditions:

$$\forall i = 1, ..., m, \quad \sum_{j=1}^{m} y_{ij} V_j = 0; \tag{II.21}$$

$$\forall i, j = 1, ..., m, \quad y_{i0} V_j^* = y_{0j} V_i. \tag{II.22}$$

As the $V_i's$ are non-zero, condition (II.22) implies that the complex y_{i0} are all zero or all non-zero for $i = 1, ..., m$.

- If they are non-zero, the same condition implies that the dimension of the space span $\{V_1, ..., V_m\}$ is one, so that $m = 1$. In this case, writing $\frac{y_{01}}{y_{10}} = a \in \mathbb{C}$, condition (II.22) is just $V_1^* = aV_1$ so that $V_1 = |a|^2 V_1$ and consequently $|a| = 1$. We write $a = e^{i\theta}$ with $\theta \in \mathbb{R}$ and we obtain (recall that we are in the case $m = 1$):

  $$H_1 = H_{\mathcal{S}} \otimes I_{\mathcal{K}} + e^{-i\theta/2} V_1 \otimes \begin{pmatrix} 0 & e^{-i\theta/2} \\ e^{i\theta/2} & 0 \end{pmatrix}.$$

  Furthermore it is straightforward that for any unitary operator W on $\mathcal{K}'$,

  $$W^* \star \begin{pmatrix} V_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \langle e_1, W^* e_1 \rangle V_1 \\ \vdots \\ \langle e_d, W^* e_1 \rangle V_1 \end{pmatrix}.$$

  Writing $a_i = \langle e_i, W^* e_1 \rangle$ for all $i = 1, ..., d$ by Lemma II.2 we obtain that H is of the form of Equation (II.17).

- Suppose that the y_{0i}'s are null. Then condition (II.21) and the fact that the V_k's are linearly independent imply that the matrix $(y_{ij})_{1 \le i,j \le m}$ is null, and consequently that $Y = 0$ (we have assume $y_{00} = 0$). It shows that in this case $\mathcal{A}(H_1) = \mathcal{B}(\mathcal{K}_1)$, which conclude the proof.

□

III The Environment Right-Action Algebra

If one is only interested in the evolution of the observables of the system, then the algebra $\mathcal{A}(U)$ may not be the most relevant. For instance, consider the unitary operators U and V on $\mathcal{H} \otimes \mathbb{C}^2$, written as block-matrices in the canonical basis of $\mathbb{C}^2$:

$$U = \begin{pmatrix} 0 & U_2 \\ U_1 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}, \tag{III.1}$$

where U_1 and U_2 are two unitary operators on $\mathcal{H}$. If U_1 and U_2 are not collinear, then it is not difficult to compute that $\mathcal{A}(U) = \mathcal{B}(\mathcal{K})$ and $\mathcal{A}(V) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad a, b \in \mathbb{C} \right\}$ which is a commutative algebra. However, computing the corresponding evolution of an observable of the system $X \in \mathcal{B}(\mathcal{H})$ in the Heisenberg picture leads to:

$$U^* (X \otimes I_{\mathcal{K}}) U = V^* (X \otimes I_{\mathcal{K}}) V.$$

Thus, from the point of view of the system, U and V have the same action. Consequently U leads to a classical action of the environment in the sense of [ADP], a fact which was not captured by the algebra $\mathcal{A}(U)$. Remark however that U and V are linked by the relation

$$U = \left[I_{\mathcal{H}} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] V. \tag{III.2}$$

We will come back on this remark and generalize it in Subsection III. 2.

In Subsection III. 1 we introduce the *Environment Right-Action Algebra* and give a first characterization of it. In Subsection III. 3, we emphasize a link between this algebra and minimal Stinespring representations of CP maps. Finally in Subsection III. 4 we give a characterization of the Environment Algebra and the Environment Right-Action Algebra in term of the spectrum of a CP map, which provide an operational way to compute those two algebras.

III. 1 Definition and first examples

To capture the idea of the action of the environment on the system we introduce the following algebra.

Definition III.1. *Let U be a unitary operator on $\mathcal{H} \otimes \mathcal{K}$. Then we call the* Environment Right-Action Algebra *the von Neumann algebra $\mathcal{A}_r(U)$ defined by*

$$\mathcal{A}_r(U) = \{U^*(f_1, g_1)U(f_2, g_2); \quad f_1, f_2, g_1, g_2 \in \mathcal{H}\}''. \tag{III.3}$$

In the same way we could have introduced the *Environment Left-Action Algebra*, in order to describe the relevant part of the environment for the evolution $UX \otimes I_{\mathcal{K}}U^*$. Most of the results that follow have their counterpart for this algebra, yet we prefer to focus on the Environment Right-Action Algebra as it corresponds to the physical situation of the Heisenberg picture of time evolution.

As for the Environment Algebra, the Environment Right-Action Algebra is easily characterized.

Proposition III.1. *Let U be a unitary operator on $\mathcal{H} \otimes \mathcal{K}$. Then $\mathcal{A}_r(U)$ is the smallest von Neumann subalgebra of $\mathcal{B}(\mathcal{K})$ such that*

$$U^* (X \otimes I_{\mathcal{K}}) U \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{A}_r(U) \text{ for all } X \in \mathcal{B}(\mathcal{H}), \tag{III.4}$$

i.e. if $\mathcal{A}$ is an other von Neumann subalgebra of $\mathcal{B}(\mathcal{K})$ such that Equation (III.4) holds, then $\mathcal{A}_r(U) \subset \mathcal{A}$. Furthermore its commutant is given by

$$\mathcal{A}_r(U)' = \{Y \in \mathcal{B}(\mathcal{K}),\ [I \otimes Y, U^* (X \otimes I_{\mathcal{K}}) U] = 0 \text{ for all } X \in \mathcal{B}(\mathcal{H})\}. \tag{III.5}$$

Proof. The proof is the same as Proposition II.3. First we prove Equality (III.5) by a direct computation. Then, if $\mathcal{A}$ is an other von Neumann algebra such that Equation (III.4) holds, one verifies immediately that $\mathcal{A}' \subset \mathcal{A}_r(U)'$, so that the result follows by the Bicommutant Theorem. □

Corollary III.1. *Let U be a unitary operator on $\mathcal{H} \otimes \mathcal{K}$. Then $\mathcal{A}_r(U)$ is a subalgebra of $\mathcal{A}(U)$.*

Proof. As $\mathcal{B}(\mathcal{H}) \otimes I_{\mathcal{K}} \subset \mathcal{B}(\mathcal{H}) \otimes \mathcal{A}(U)$ and $U, U^* \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{A}(U)$, we have $U^* (\mathcal{B}(\mathcal{H}) \otimes I_{\mathcal{K}}) U \subset \mathcal{B}(\mathcal{H}) \otimes \mathcal{A}(U)$. By the characterization of $\mathcal{A}_r(U)$, the corollary holds. □

As in the case of the Environment Algebra, commutative Right-Action Algebras and the Spontaneous Emission are the two main examples. The commutative case will be studied as a corollary of Theorem III.1, so that we only treat the case of the Spontaneous Emission here.

Example of the Spontaneous Emission: Consider again the unitary operator U_{se} given by Equation (I.1), with $\theta \notin 2\pi\mathbb{Z}$. We show that $\mathcal{A}_r(U) = \mathcal{A}(U) = \mathcal{B}(\mathbb{C}^2)$, so that the Environment Right-Action is also purely quantum.

To check this, we show that there exists a pure state $|\Omega\rangle\langle\Omega|$ of $\mathcal{K}$ such that $\mathcal{L}_{|\Omega\rangle\langle\Omega|}$ is not of the form (II.6). If it were the case, $\mathcal{L}_\omega$ would be trace-preserving. Let (e_0, e_1) be the canonical basis of $\mathbb{C}^2$. Then

$$\begin{aligned}\mathcal{L}_{|e_0\rangle\langle e_0|}(|e_0\rangle\langle e_0|) &= \begin{pmatrix}1 & 0\\ 0 & \cos\theta\end{pmatrix}|e_0\rangle\langle e_0|\begin{pmatrix}1 & 0\\ 0 & \cos\theta\end{pmatrix}\\ &+ \begin{pmatrix}0 & 0\\ \sin\theta & 0\end{pmatrix}|e_0\rangle\langle e_0|\begin{pmatrix}0 & \sin\theta\\ 0 & 0\end{pmatrix}\\ &= \begin{pmatrix}1 & 0\\ 0 & \sin^2\theta\end{pmatrix}.\end{aligned}$$

It is clear that whenever $\theta \notin 2\pi\mathbb{Z}$, $\mathrm{Tr}\left[\mathcal{L}_{|e_0\rangle\langle e_0|}(|e_0\rangle\langle e_0|)\right] \neq 1$.
Consequently, we obtain the announced result:

$$\mathcal{A}_r(U_{\mathrm{se}}) = \mathcal{B}(\mathbb{C}^2). \tag{III.6}$$

Note that with this method we also directly prove that $\mathcal{A}(U_{\mathrm{se}}) = \mathcal{B}(\mathbb{C}^2)$, as by corollary III.1 $\mathcal{A}_r(U) \subset \mathcal{A}(U)$.

III. 2 Link between $\mathcal{A}(U)$ and $\mathcal{A}_r(U)$

As we remarked before, there is a relation between two unitary operators that have the same action on the system. This point is emphasized in the following lemma (which is originally from [DNP15]).

Lemma III.1. *Let U and V be two unitary operators on $\mathcal{H} \otimes \mathcal{K}$. Suppose that for all $X \in \mathcal{B}(\mathcal{H})$,*

$$V^* \left(X \otimes I_{\mathcal{K}}\right) V = U^* \left(X \otimes I_{\mathcal{K}}\right) U \tag{III.7}$$

Then there exists a unitary operator W on $\mathcal{K}$ such that

$$V = \left(I_{\mathcal{H}} \otimes W\right) U. \tag{III.8}$$

Proof. From Equation (III.7) we obtain that for all $X \in \mathcal{B}(\mathcal{H})$, $X \otimes I_{\mathcal{K}} = (VU^*)^* \left(X \otimes I_{\mathcal{K}}\right) (VU^*)$ and then $[VU^*, X \otimes I_{\mathcal{K}}] = 0$. This in turn implies the existence of a unitary operator W on $\mathcal{K}$ such that $VU^* = I_{\mathcal{H}} \otimes W$. Clearly W is unitary. □

We call $\mathcal{R}_r(U)$ the class of U for this relation, that is, $V \in \mathcal{R}_r(U)$ if and only if Equality (III.7) holds. It is straightforward that it is an equivalence relation and, because of Proposition II.3, that every element in this class share the same Environment Right-Action Algebra. That is, for all $V \in \mathcal{R}_r(U)$, we have $\mathcal{A}_r(V) = \mathcal{A}_r(U)$. Moreover, because of Corollary III.1 for all $V \in \mathcal{R}_r(U)$ one has $\mathcal{A}_r(U) \subset \mathcal{A}(V)$. Consequently,

$$\mathcal{A}_r(U) \subset \bigcap_{V \in \mathcal{R}_r(U)} \mathcal{A}(V).$$

A natural question now is whether the equality holds. To answer this question we show under some hypotheses the following: there exists an element in the class $\mathcal{R}_r(U)$ whose Environment Algebra and Environment Right-Action Algebra coincide. This result also implies that we can reduce the study of $\mathcal{A}_r(U)$ to the one of $\mathcal{A}(V)$ for a specific $V \in \mathcal{R}_r(U)$.

Theorem III.1. *Suppose that $\mathcal{H}$ is finite dimensional and that $\mathcal{A}_r(U)$ is a type I von Neumann algebra. Then:*

1. *there exists a unitary operator $V \in \mathcal{R}_r(U)$ such that:*

$$V \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{A}_r(U) \tag{III.9}$$

and consequently

$$\mathcal{A}(V) = \mathcal{A}_r(V) = \mathcal{A}_r(U). \tag{III.10}$$

2. *If $V_1, V_2 \in \mathcal{R}_r(U)$ both satisfy Equation (III.10), then $V_1 V_2^* \in I_{\mathcal{H}} \otimes \mathcal{A}_r(U)$.*

At least when $\mathcal{K}$ is finite dimensional, for all $f, g \in \mathcal{H}$, one can write the polar decomposition of the operator $U(f,g)$ as

$$U(f,g) = W\sqrt{U(f,g)^* U(f,g)}, \tag{III.11}$$

where W is a unitary operator on $\mathcal{K}$ that depends on f and g. If it were the case, $(I_{\mathcal{H}} \otimes W)U$ would be the wanted element of the class. Theorem III.1 can thus be interpreted as a kind of polar decomposition of the operator U with respect to the environment, where the operator $\sqrt{U(f,g)^* U(f,g)}$ is replaced by an element of $\mathcal{A}_r(U)$. A similar result can be found in [DNP15].

Proof. The proof is based on the central decomposition of $\mathcal{A}_r(U)$ as a type I von Neumann algebra [Tak79]. The space $\mathcal{K}$ has a direct integral representation $\mathcal{K} = \int_A^{\oplus} \mathcal{K}_\alpha \, \mathbb{P}(d\alpha)$ in the sense that there exists a family of Hilbert space $(\mathcal{K}_\alpha)_{\alpha\in A}$ and for any $\psi \in \mathcal{K}$ there exists a map $\alpha \in A \mapsto \psi_\alpha$ such that

$$\langle \psi, \phi \rangle = \int_A^{\oplus} \langle \psi_\alpha, \phi_\alpha \rangle \mathbb{P}(d\alpha).$$

The von Neumann algebra $\mathcal{A}_r(U)$ has a central decomposition

$$\mathcal{A}_r(U) = \int_A^{\oplus} \mathcal{B}(\mathcal{K}_\alpha)\mathbb{P}(d\alpha)$$

in the sense that for any $Y \in \mathcal{A}_r(U)$ there exists a map $\alpha \in A \mapsto Y_\alpha \in \mathcal{B}(\mathcal{K}_\alpha)$ such that $(Y\psi)_\alpha = Y_\alpha \psi_\alpha$ for almost all α. Then for all $X \in \mathcal{B}(\mathcal{H})$,

$$U^* (X \otimes I_{\mathcal{K}}) U = \int_{\alpha\in A} T_\alpha(X)\mathbb{P}(d\alpha),$$

where T_α are linear maps from $\mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K}_\alpha)$. We will ignore for simplicity all issues related with measurability in the following constructions. Let us fix $\alpha \in A$. It is a simple computation to show that T_α is a $\star$-morphism that preserves unity: it is a representation of $\mathcal{B}(\mathcal{H})$ into $\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K}_\alpha)$, which furthermore is normal. The goal is to write it in the form $T_\alpha(X) = V_\alpha^* (X \otimes I_{\mathcal{K}_\alpha}) V_\alpha$ for some unitary operator V_α on $\mathcal{H} \otimes \mathcal{K}_\alpha$. As it is a normal representation of $\mathcal{H}$ on $\mathcal{H} \otimes \mathcal{K}_\alpha$, $\mathcal{H} \otimes \mathcal{K}_\alpha$ can be decomposed as the direct sum of some orthogonal spaces $\mathcal{K}_{\alpha,\beta}$ such that (see [Att] or [Fag99] for instance)

$$T_\alpha(X) = \sum_{\beta\in B} V_{\alpha,\beta} X V_{\alpha,\beta}^*,$$

where the $V_{\alpha,\beta}$'s are unitary operators from $\mathcal{H}$ into $\mathcal{K}_{\alpha,\beta}$ and B is a finite or countable set. As $\mathcal{H}$ is finite dimensional, so are the $\mathcal{K}_{\alpha,\beta}$ and they have the same dimension than $\mathcal{H}$. It implies that either B is a finite set with $\dim \mathcal{K}_\alpha$ elements if $\mathcal{K}_\alpha$ is finite dimensional, either B is a countable infinite set if $\mathcal{K}_\alpha$ is infinite dimensional.

Either cases, let $(e_\beta)_{\beta\in B}$ be an orthonormal basis of $\mathcal{K}_\alpha$ and define the operator V_α in $\mathcal{B}(\mathcal{H} \otimes \mathcal{K}_\alpha)$ by the formula

$$V_\alpha^* (f \otimes e_\beta) = V_{\alpha,\beta} f, \text{ for all } f \in \mathcal{H} \text{ and } \beta \in B.$$

Then one has

$$\begin{aligned} V_\alpha^* V_\alpha &= \sum_{\beta\in B} V_\alpha^* (I_{\mathcal{H}} \otimes |e_\beta\rangle\langle e_\beta|) V_\alpha \\ &= \sum_{\beta\in B} V_{\alpha,\beta} I_{\mathcal{H}} V_{\alpha,\beta}^* \\ &= \sum_{\beta\in B} I_{\mathcal{H}\otimes\mathcal{K}_{\alpha,\beta}} = I_{\mathcal{H}\otimes\mathcal{K}_\alpha}. \end{aligned}$$

Similarly $V_\alpha V_\alpha^* = I_{\mathcal{H}\otimes\mathcal{K}_\alpha}$, so that V_α is a unitary operator. Moreover, for all $f, g \in \mathcal{H}$,

$$\begin{aligned} V_\alpha^* \left(|f\rangle\langle g| \otimes I_{\mathcal{K}_\alpha}\right) V_\alpha &= \sum_{\beta\in B} V_{\alpha,\beta}|f\rangle\langle g|V_{\alpha,\beta}^* \\ &= T_\alpha(|f\rangle\langle g|). \end{aligned}$$

This formula extends to all $\mathcal{B}(\mathcal{H})$ by strong continuity.
Now write $V = \int_{\alpha\in A} V_\alpha \mathbb{P}(d\alpha)$. By construction it is a unitary operator in $\mathcal{B}(\mathcal{H}) \otimes \mathcal{A}_r(U)$ and for all $X \in \mathcal{B}(\mathcal{H})$,

$$U^* \left(X \otimes I_{\mathcal{K}}\right) U = V^* \left(X \otimes I_{\mathcal{K}}\right) V.$$

Now by Lemma II.1, we have $\mathcal{A}_r(V) = \mathcal{A}_r(U)$ and, as $V \in \mathcal{A}_r(V)$, we have $\mathcal{A}(V) \subset \mathcal{A}_r(V)$ by Proposition II.3. Consequently $\mathcal{A}_r(V) = \mathcal{A}(V)$, so point 1) in the Theorem is proved. Let us prove point 2). If V_1 and V_2 both satisfy Equation (III.10), then, again by Lemma II.1, there exists a unitary operator W on $\mathcal{K}$ such that $V_1 = I_{\mathcal{H}}\otimes W V_2$. Then $V_1V_2^* = I_{\mathcal{H}}\otimes W$, so that $V_1V_2^* \in I_{\mathcal{H}} \otimes \mathcal{B}(\mathcal{K})$. As $V_1, V_2 \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{A}_r(U)$, necessarily $W \in \mathcal{A}_r(U)$ and the result follows. □

We obtain the following characterization for commutative Environment Right-Action Algebra for a finite dimensional environment:

Corollary III.2. *Suppose that $\mathcal{H}$ and $\mathcal{K}$ are finite dimensional, with dimension N and d respectively. Let U be a unitary operator on $\mathcal{H} \otimes \mathcal{K}$. Then $\mathcal{A}_r(U)$ is commutative if and only if there exist two orthonormal basis (φ_i) and (ψ_i) of $\mathcal{K}$ and unitary operators $U_1, ..., U_d$ on $\mathcal{H}$ such that*

$$U = \sum_{i=1}^{d} U_i \otimes |\varphi_i\rangle\langle\psi_i|. \tag{III.12}$$

Proof. We first show that (i)⇒ (ii). Because of Theorem III.1 there exists a unitary operator W on $\mathcal{K}$ such that
$\mathcal{A}\left(I_{\mathcal{H}} \otimes W\ U\right) = \mathcal{A}_r(U)$. Consequently, as $\mathcal{A}_r(U)$ is commutative, there exist an orthonormal basis (ψ_i) of $\mathcal{K}$ and unitary operators $U_1, ..., U_d$ on $\mathcal{H}$ such that

$$I_{\mathcal{H}} \otimes WU = \sum_{i=1}^{d} U_i \otimes |\psi_i\rangle\langle\psi_i|.$$

Write $\varphi_i = W^*\psi_i$ for all i. Then (φ_i) is an orthonormal basis of $\mathcal{K}$ and Equation (III.12) holds. □

III. 3 Minimal Stinespring representation

There is a link between the structure of $\mathcal{A}_r(U)$ and dilation of CP maps obtained via U. In particular we obtain a sufficient condition so that $\mathcal{A}_r(U) = \mathcal{B}(\mathcal{K})$. First we recall some definitions.

Definition III.2. *Let $\mathcal{L}$ be a normal CP map on $\mathcal{H}$, and V be an isometry from $\mathcal{H}$ to $\mathcal{H} \otimes \mathcal{K}$, such that, for all $X \in \mathcal{B}(\mathcal{H})$, we have:*

$$\mathcal{L}(X) = V^* (X \otimes I_{\mathcal{K}}) V.$$

The couple $(\mathcal{H} \otimes \mathcal{K}, V)$ is called a Stinespring representation *of $\mathcal{L}$. Furthermore, this representation is defined to be* minimal *if the set*

$$\mathcal{V} = \{(X \otimes I_{\mathcal{K}}) V f, \ X \in \mathcal{B}(\mathcal{H}), f \in \mathcal{H}\} \tag{III.13}$$

is total in $\mathcal{H} \otimes \mathcal{K}$.

By Stinespring Theorem ([Sti55]) every normal CP map has a minimal Stinespring representation of this form (see [Att] and [Fag99]). Furthermore, if $(\mathcal{H} \otimes \mathcal{K}_1, V_1)$ and $(\mathcal{H} \otimes \mathcal{K}_2, V_2)$ are two minimal Stinespring representations of $\mathcal{L}$, then there exists a unitary operator W from $\mathcal{K}_1$ to $\mathcal{K}_2$ such that for all $X \in \mathcal{B}(\mathcal{H})$:

$$(I_{\mathcal{H}} \otimes W) V_1 = V_2.$$

Now we come back to the case of a unitary operator U on the bipartite space $\mathcal{H} \otimes \mathcal{K}$. For every pure state $\psi \in \mathcal{K}$ define the isometry $U_\psi : \mathcal{H} \to \mathcal{H} \otimes \mathcal{K}$ by

$$U_\psi : f \mapsto U f \otimes \psi. \tag{III.14}$$

We remark that the couple $(\mathcal{H} \otimes \mathcal{K}, U_\psi)$ is a Stinespring representation of the CP map $\mathcal{L}_{|\psi\rangle\langle\psi|}$ defined by

$$\mathcal{L}_{|\psi\rangle\langle\psi|} : X \in \mathcal{B}(\mathcal{H}) \to \mathrm{Tr}_{|\psi\rangle\langle\psi|} [U^* X \otimes I_{\mathcal{K}} U]. \tag{III.15}$$

A direct application of Lemma III.1 is that $V \in \mathcal{R}_r(U)$ if and only if $\mathrm{Tr}_{|\psi\rangle\langle\psi|} [V^* \cdot \otimes I_{\mathcal{K}} V] = \mathcal{L}_{|\psi\rangle\langle\psi|}(\cdot)$ for all pure state $\psi \in \mathcal{K}$ (which was the initial Lemma in [DNP15]). The following proposition completes this statement.

Proposition III.2. *The property for $(\mathcal{H}\otimes\mathcal{K},U_\psi)$ to give a minimal Stinespring representation of $\mathcal{L}_{|\psi\rangle\langle\psi|}$ is a property of the class $\mathcal{R}_r(U)$, i.e. if $(\mathcal{H}\otimes\mathcal{K},U_\psi)$ is minimal, then for all $V\in\mathcal{R}_r(U)$, the Stinespring representation $(\mathcal{H}\otimes\mathcal{K},V_\psi)$ is also minimal.*

Proof. Let $\psi\in\mathcal{K}$ be a pure state. For all unitary operators V on $\mathcal{K}$, we write

$$\mathcal{V}_\psi(V)=\{(X\otimes I)Vf\otimes\psi,\ X\in\mathcal{B}(\mathcal{H}),f\in\mathcal{H}\}.$$

Suppose that $(\mathcal{H}\otimes\mathcal{K},U_\psi)$ is a minimal Stinespring representation of $\mathcal{L}_{|\psi\rangle\langle\psi|}$. By definition it means that $\mathcal{V}_\psi(U)$ is total in $\mathcal{H}\otimes\mathcal{K}$. Take $V\in\mathcal{R}_r(U)$. Because of Lemma III.1, there exists a unitary operator W on $\mathcal{K}$ such that $V=(I_\mathcal{H}\otimes W)U$. Let x be an element of $\mathcal{V}_\psi(V)^\perp$. Then for all $X\in\mathcal{B}(\mathcal{H})$ and $f\in\mathcal{H}$:

$$0=\langle x,(X\otimes W)U(f\otimes\psi)\rangle=\langle(I\otimes W^*)x,(X\otimes I)U(f\otimes\psi)\rangle.$$

As $\mathcal{V}_\psi(U)$ is total in $\mathcal{H}\otimes\mathcal{K}$, this implies that $(I\otimes W^*)x=0$, and consequently $x=0$. □

Let $\mathcal{A}$ be a von Neumann subalgebra of $\mathcal{B}(\mathcal{K})$. We recall that a vector $\psi\in\mathcal{K}$ is cyclic for $\mathcal{A}$ if $\overline{\mathcal{A}\psi}=\mathcal{K}$. There does not always exist a cyclic vector. For example, a commutative algebra $\mathcal{A}$ has a cyclic vector if and only if $\mathcal{A}=\mathcal{A}'$ ([RS80]).

We now suppose that $\mathcal{H}$ is finite dimensional and that $\mathcal{A}_r(U)$ is a type I von Neumann algebra in order to be in position to apply Theorem III.1

Proposition III.3. *If $(\mathcal{H}\otimes\mathcal{K},U_\psi)$ is a minimal Stinespring representation of $\mathcal{L}_{|\psi\rangle\langle\psi|}$, then ψ is a cyclic vector for $\mathcal{A}_r(U)$.*

Proof. First, by Theorem III.1, there exists $V\in\mathcal{R}_r(U)$ such that $\mathcal{A}(V)=\mathcal{A}_r(U)$. Recall that $\mathcal{A}(V)$ is generated by the elements $V(f,g)$ defined by Equation (II.8), $f,g\in\mathcal{H}$. Consequently the subset of $\mathcal{K}$

$$\mathcal{K}_\psi(V)=\{V(f,g)\psi,\ f,g\in\mathcal{H}\},$$

is a subset of $\mathcal{A}_r(U)\psi$. Now if $(\mathcal{H}\otimes\mathcal{K},U_\psi)$ is a minimal Stinespring representation of $\mathcal{L}_{|\psi\rangle\langle\psi|}$, it is also the case for $(\mathcal{H}\otimes\mathcal{K},V_\psi)$ by Proposition III.2, so that $\mathcal{V}_\psi(V)$ is a total set. Let us show that $\mathcal{K}_\psi(V)$ is also a total set, to complete the proof. Take $\varphi\in\mathcal{K}_\psi(V)^\perp$. Then for all $f,g\in\mathcal{H}$ we have:

$$\begin{aligned}0&=\langle\varphi,V(f,g)\psi\rangle\\&=\operatorname{Tr}\left[|\psi\rangle\langle\varphi|V(f,g)\right]\\&=\operatorname{Tr}\left[(|g\rangle\langle f|\otimes|\psi\rangle\langle\varphi|)\,V\right]\\&=\langle f\otimes\varphi,Vg\otimes\psi\rangle.\end{aligned}$$

Consequently, for all $f \in \mathcal{H}$, $f \otimes \varphi \in \mathcal{V}_\psi^\perp = \{0\}$. Thus $\varphi = 0$, which concludes the proof. □

The following corollary is now straightforward.

Corollary III.3. *If for all $\psi \in \mathcal{K}$, $(\mathcal{H} \otimes \mathcal{K}, U_\psi)$ is a minimal Stinespring representation of $\mathcal{L}_{|\psi\rangle\langle\psi|}$, then $\mathcal{A}_r(U) = \mathcal{B}(\mathcal{K})$.*

Proof. It just comes from the fact that if $\mathcal{A}$ is a subalgebra of $\mathcal{B}(\mathcal{K})$ such that $\mathcal{A}\psi$ is total for all $\psi \in \mathcal{K}$, then $\mathcal{A} = \mathcal{B}(\mathcal{K})$. □

III. 4 A characterization of the algebras in terms of the spectrum of a CP map

In this Subsection, $\mathcal{H}$ and $\mathcal{K}$ are finite dimensional Hilbert spaces, of dimension N and d respectively. We give a characterization of the algebras $\mathcal{A}(U)$ and $\mathcal{A}_r(U)$ in terms of the spectrum of a specific CP map $\mathcal{L}$ acting on $\mathcal{B}(\mathcal{K})$. This CP map describes the evolution of the states of $\mathcal{K}$, i.e. density matrices on $\mathcal{K}$, whenever the state of the system is the maximally mixed state $\frac{1}{N} I_{\mathcal{H}}$. Thus $\mathcal{L}$ is given by the following formula, where $X \in \mathcal{B}(\mathcal{K})$:

$$\mathcal{L}(X) = \mathrm{Tr}_{\mathcal{H}}\left[U\left(\frac{1}{N} I_{\mathcal{H}} \otimes X\right)U^*\right]. \tag{III.16}$$

We will also need the adjoint of $\mathcal{L}$ for the Hilbert-Schmidt scalar product on $\mathcal{B}(\mathcal{K})$ $(X, Y) \mapsto \mathrm{Tr}[X^* Y]$:

$$\mathcal{L}^*(X) = \mathrm{Tr}_{\mathcal{H}}\left[U^*\left(\frac{1}{N} I_{\mathcal{H}} \otimes X\right)U\right]. \tag{III.17}$$

Our goal is to give an explicit way to compute the algebras, by relating them to some eigenspaces. Our result is the following:

Theorem III.2. *Suppose that $\mathcal{H} \approx \mathbb{C}^N$ and $\mathcal{K} \approx \mathbb{C}^d$. Then:*

1) $\mathcal{A}(U)'$ is the eigenspace of both $\mathcal{L}$ and $\mathcal{L}^*$, associated to the eigenvalue 1;

2) $\mathcal{A}_r(U)'$ is the right-singularspace of $\mathcal{L}$ associated to the right-eigenvalue 1.

The proof of Theorem III.2 will make use of Lemma III.2 below, in which we use the notion of von Neumann entropy, defined by

$$\mathcal{S}(\omega) = -\mathrm{Tr}[\phi(\omega)]. \tag{III.18}$$

where ϕ is the real and operator-monotone function $\phi : x \mapsto x \log x$, defined on $[0, 1]$.

Lemma III.2. *For all density matrix ω on $\mathcal{K}$,*

$$\mathcal{S}(\omega) \leq \mathcal{S}(\mathcal{L}(\omega)), \tag{III.19}$$

i.e. $\mathcal{L}$ is entropy increasing. Furthermore equality holds if and only if

$$U\left(\frac{1}{N}I_{\mathcal{H}} \otimes \omega\right)U^* = \frac{1}{N}I_{\mathcal{H}} \otimes \mathcal{L}(\omega). \tag{III.20}$$

Inequality (III.19) is not new. In fact $\mathcal{L}$ is a doubly stochastic CP map (i.e. it is identity and trace preserving) and basic results in majorization theory imply that a CP map verify Inequality (III.19) if and only if it is doubly stochastic ([MOA11]). However, as we are interested in the equality case, it is necessary for us to prove it.

Proof of Lemma III.2. We use the sub-additivity property of the von Neumann entropy: if ρ is a density matrix on $\mathcal{H} \otimes \mathcal{K}$, then

$$\mathcal{S}(\rho) \leq \mathcal{S}(\mathrm{Tr}_{\mathcal{K}}[\rho]) + \mathcal{S}(\mathrm{Tr}_{\mathcal{H}}[\rho]),$$

with equality if and only if $\rho = \mathrm{Tr}_{\mathcal{K}}[\rho] \otimes \mathrm{Tr}_{\mathcal{H}}[\rho]$. Applying this to the density matrix $U\left(\frac{1}{N}I_{\mathcal{H}} \otimes \omega\right)U^*$, we get:

$$\begin{aligned}\log N + \mathcal{S}(\omega) &\leq \mathcal{S}\left(U\left(\frac{1}{N}I_{\mathcal{H}} \otimes \omega\right)U^*\right)\\ &\leq \mathcal{S}\left(\mathrm{Tr}_{\mathcal{K}}[U\left(\frac{1}{N}I_{\mathcal{H}} \otimes \omega\right)U^*]\right) + \mathcal{S}(\mathcal{L}(\omega))\\ &\leq \log N + \mathcal{S}(\mathcal{L}(\omega)),\end{aligned}$$

where we use the fact that the entropy of any density matrix on $\mathcal{H}$ is lower than $\log N$, with equality for the maximally mixed state $\frac{1}{N}I_{\mathcal{H}}$. Consequently, we obtain the desired inequality:

$$\mathcal{S}(\omega) \leq \mathcal{S}(\mathcal{L}(\omega)).$$

The equality case above means that equality holds for the sub-additivity of the von Neumann entropy, which implies that there exist density matrices ρ on $\mathcal{H}$ and ω' on $\mathcal{K}$ such that $U\left(\frac{1}{N}I_{\mathcal{H}} \otimes \omega\right)U^* = \rho \otimes \omega'$. Taking the partial trace with respect to $\mathcal{K}$ then $\mathcal{H}$, we see that $\rho = \frac{1}{N}I_{\mathcal{H}}$ and that $\omega' = \mathcal{L}(\omega)$, which end the proof. □

Proof of Theorem III.2. The idea behind the proof of each part of the theorem is the same. We start with the following computation. For all $X \in \mathcal{B}(\mathcal{K})$; we have

$$\begin{aligned} X \in \mathcal{A}'(U) &\Leftrightarrow [I_{\mathcal{H}} \otimes X, U] = 0 \\ &\Leftrightarrow U\left(\frac{1}{N} I_{\mathcal{H}} \otimes X\right) U^* = \frac{1}{N} I_{\mathcal{H}} \otimes X \\ &\qquad \text{and} \quad U^*\left(\frac{1}{N} I_{\mathcal{H}} \otimes X\right) U = \frac{1}{N} I_{\mathcal{H}} \otimes X \\ &\Rightarrow \mathcal{L}(X) = X \text{ and } \mathcal{L}^*(X) = X \\ &\Rightarrow X \text{ is an eigeinvector of} \\ &\qquad \mathcal{L} \text{ associated to the eigeinvalue } 1. \end{aligned}$$

In the same way, for all $X \in \mathcal{B}(\mathcal{K})$

$$\begin{aligned} X \in \mathcal{A}_r'(U) &\Leftrightarrow [I_{\mathcal{H}} \otimes X, U^*(Y \otimes I_{\mathcal{K}})] = 0 \quad \forall Y \in \mathcal{B}(\mathcal{K}) \\ &\Leftrightarrow [U(I_{\mathcal{H}} \otimes X) U^*, Y \otimes I_{\mathcal{K}}] = 0 \quad \forall Y \in \mathcal{B}(\mathcal{K}) \\ &\Leftrightarrow \text{ there exists } X' \in \mathcal{B}(\mathcal{K}) \text{ such that:} \\ &\qquad U\left(\frac{1}{N} I_{\mathcal{H}} \otimes X\right) U^* = \frac{1}{N} I_{\mathcal{H}} \otimes X' \\ &\Rightarrow \text{ there exists } X' \in \mathcal{B}(\mathcal{K}) \text{ such that:} \\ &\qquad \mathcal{L}(X) = X' \text{ and } \mathcal{L}^*(X') = X \\ &\Rightarrow X \text{ is a right-singularvector of} \\ &\qquad \mathcal{L} \text{ associated to the singularvalue } 1. \end{aligned}$$

In order to complete the proof, we need to show the converse. We do that for $\mathcal{A}_r'(U)$ only, as it is the same for $\mathcal{A}(U)'$.

First remark that $\mathcal{L}(I_{\mathcal{K}}) = \mathcal{L}^*(I_{\mathcal{K}}) = I_{\mathcal{K}}$, so that $I_{\mathcal{K}}$ is always an eigenvector of $\mathcal{L}$ and $\mathcal{L}^*$ for the eigenvalue 1 (and consequently a left and right singularvector for the singularvalue 1). Furthermore, if $\mathcal{L}(X) = X'$ and $\mathcal{L}^*(X') = X$ for some $X, X' \in \mathcal{B}(\mathcal{K})$, than $\mathcal{L}(X^*) = X'^*$ and $\mathcal{L}^*(X'^*) = X^*$. Consequently, the right-singularspace of $\mathcal{L}$ associated to the right-eigenvalue 1 is a system operator, that is a norm-closed $*$-stable subspace of $\mathcal{B}(\mathcal{K})$. In particular, it is generated by its positive elements, so that we only need to prove the result for density matrices.

In view of Lemma III.2, we only need to prove that for all density matrices ω such that $\omega = \mathcal{L}^* \circ \mathcal{L}(\omega)$,

$$\mathcal{S}(\omega) = \mathcal{S}(\mathcal{L}(\omega)).$$

Applying again Lemma III.2 both for U^* and U, we get

$$\mathcal{S}(\mathcal{L}^* \circ \mathcal{L}(\omega)) = \mathcal{S}(\omega) \leq \mathcal{S}(\mathcal{L}(\omega)) \leq \mathcal{S}(\mathcal{L}^* \circ \mathcal{L}(\omega)),$$

which shows the desired equality. □

Chapter 2

Article: Classical and Quantum parts of the environment for the quantum Langevin Equation

Abstract:

Among quantum Langevin equations describing the unitary time evolution of a quantum system in contact with a quantum bath, we completely characterize those equations which are actually driven by classical noises. The characterization is purely algebraic, in terms of the coefficients of the equation. In a second part, we consider general quantum Langevin equations and we prove that they can always be split into a maximal commutative part and a purely quantum one.

I Introduction

Since the construction of Quantum Stochastic Calculus and the corresponding quantum stochastic differential equations (quantum Langevin Equations) on the symmetric Fock space ([HP84]), it is well-known that both classical and quantum noises could coexist in the equation. The framework is designed for quantum noises, however some classical noises can also appear with some particular combinations of the quantum noises (see S. Attal's lecture in [AJP06]). This fact was the starting point of the recent articles [ADP13] and [ADP], where the authors characterized all the possible classical processes that can emerge in such quantum Langevin equations: they are the complex normal martingales in $\mathbb{C}^n$; up to a unitary transform of $\mathbb{C}^n$ they are combinations of independent Wiener processes and Poisson processes in different directions of the space.

Let us be more explicit with some simple examples. The quantum system state space is the Hilbert space $\mathcal{H}$, whereas the quantum heat bath is represented by quantum noises $da^i_j(t)$ (see Subsection II. 1 for the notations) on the symmetric Fock space $\Phi = \Gamma_s(L^2(\mathbb{R}^+;\mathbb{C}^n))$. In the simplest case where $n = 1$, the joint evolution between the system and its environment can be described by a one-parameter family of unitary operators (U_t), solving a quantum Langevin equation:

$$dU_t = \left(iH - \frac{1}{2}L^*L\right) U_t\,dt + L\,U_t\,da^0_1(t) - L^*S\,U_t\,da^1_0(t) + (S-I)\,U_t\,da^1_1(t)\,. \tag{I.1}$$

where H, L and S are operators on $\mathcal{H}$ such that $H = H^*$ and S is a unitary operator.

In the particular case where $S = I$ and $L = -L^*$, Equation (I.1) becomes

$$dU_t = \left(iH + \frac{1}{2}L^2\right) U_t\,dt + L\,U_t\left(da^0_1(t) + da^1_0(t)\right)\,. \tag{I.2}$$

But it is well-known that the combinations of operators $B_t = a^0_1(t) + a^1_0(t)$ are naturally isomorphic to the multiplication operators by the Brownian motion on its canonical space. Hence Equation (I.2) is actually a Brownian motion driven unitary evolution:

$$dU_t = \left(iH + \frac{1}{2}L^2\right) U_t\,dt + L\,U_t\,dB_t\,.$$

Note that the conditions on H and S are the most general ones for a Brownian motion driven operator-valued equation to give unitary solutions.

In the other particular case where $L = \rho\,(S - I)$, for some $\rho > 0$, Equation (I.1) becomes

$$dU_t = \left(iH - \frac{1}{2}L^*L\right) U_t\,dt + L\,U_t\left(da^0_1(t) + da^1_0(t) + \frac{1}{\rho}\,da^1_1(t)\right)\,. \tag{I.3}$$

The combinations of operators $X_t = a^0_1(t) + a^1_0(t) + 1/\rho\; a^1_1(t)$ are naturally isomorphic to the multiplication operators by the compensated Poisson process with intensity ρ^2 and jumps $1/\rho$ on its canonical space. Hence Equation (I.3) is actually a Poisson process driven unitary evolution:

$$dU_t = \left(iH - \frac{1}{2}L^*L\right) U_t\,dt + L\,U_t\,dX_t\,.$$

Note that the conditions above on the coefficients are the most general ones for a Poisson process driven operator-valued equation to give unitary solutions.

In general, it can be shown that for the von Neumann algebra generated by $\mathcal{B}(\mathcal{H})\otimes I_\Phi$ and $\{U_t\}_{t\geq 0}$ to be of the form $\mathcal{B}(\mathcal{H})\otimes\mathcal{A}$ with $\mathcal{A}$ commutative, one of the two following conditions must holds:

– either $S = I_{\mathcal{H}}$ and there exists $\theta \in \mathbb{R}$ such that $L^* = e^{i\theta}L$,

– or there exists a complex number λ such that $L = \lambda(S - I_{\mathcal{H}})$.

The first motivation of this paper is to give a criteria on the unitary solution of a quantum Langevin Equation so that it is driven by classical noises, generalizing the preceding remark.

On the other hand, some evolutions are understood to be typically quantum or non-commutative, although there is no clear definition of what it means. This is for instance the case for the spontaneous emission, where the evolution is given by the unitary solution of the following quantum Langevin Equation:

$$dU_t = -\frac{1}{2}V^*V\,U_t\,dt + V\,U_t\,da_1^0(t) - V^*\,U_t\,da_0^1(t), \tag{I.4}$$

where

$$V = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

In a second part of the article we show that any quantum Langevin equation can be split into two parts: a maximal commutative one (that is, driven by a classical noise and maximal in dimension) and a purely quantum one (that is, which contains no classical part whatsoever).

The article is structured as follows. In Section 2 we recall a few notations concerning quantum noises, quantum Langevin equations and their probabilistic interpretations. We discuss the notion of *change of noise*, that is, the effect on the quantum noise of a change of basis. We recall the definition and the main properties of the *noise algebra*, as defined in [Bar15].

Section 3 is devoted to our main result: a complete characterization of those quantum Langevin equations that give rise to a commutative noise algebra. The characterization is given both in algebraic properties of the coefficients and in probabilistic interpretations of the classical noises appearing in the equation. As a corollary we obtain a characterization of the corresponding Lindblad generators.

In Section 4, we are back to general quantum Langevin equations and we prove that they all admit a splitting into a maximal commutative part and a purely quantum one (in the sense that it admits no commutative subspace). We end this section and the article with some discussion and examples.

II The Noise Algebra

II. 1 Notations

Let us recall here a few notations concerning quantum noises.

Let $\mathcal{K}$ be a finite dimensional Hilbert space of dimension d. We put $\Lambda = \{1, ..., d\}$ and we consider a fixed orthonormal basis $(e_i)_{i\in\Lambda}$ of $\mathcal{K}$. We denote by $\Phi = \Gamma_s\left(L^2(\mathbb{R}^+, \mathcal{K})\right)$ the symmetric Fock space over $L^2(\mathbb{R}^+, \mathcal{K})$. We are given ourselves another auxiliary Hilbert space $\mathcal{H}$ which represents the "small system" state space in quantum Langevin equations. We put $\Psi = \mathcal{H} \otimes \Phi$.

On the Fock space Φ we consider the usual creation operators $A^\dagger(f)$ and annihilation operators $A(f)$, for all $f \in L^2(\mathbb{R}^+, \mathcal{K})$; we also consider the differential second quantization operators $d\Gamma(H)$, for all self-adjoint operator H on $L^2(\mathbb{R}^+, \mathcal{K})$. The quantum noises $a^i_j(t)$, $i, j \in \Lambda \cup \{0\}$, are then defined as follows:

$$
\begin{aligned}
a^0_i(t) &= A^\dagger\left(\mathbb{1}_{[0,t]}\,|e_i\rangle\right),\\
a^i_0(t) &= A\left(\mathbb{1}_{[0,t]}\,\langle e_i|\right)\\
a^i_j(t) &= d\Gamma\left(|e_j\rangle\langle e_i|\mathcal{M}_{\mathbb{1}_{[0,t]}}\right),
\end{aligned}
$$

where $\mathbb{1}_{[0,t]}$ is the the usual indicator function of the interval $[0, t]$, where $\mathcal{M}_{\mathbb{1}_{[0,t]}}$ is the multiplication operator by $\mathbb{1}_{[0,t]}$ and where we used the usual “bra" and “ket" notations for vectors and linear forms on $\mathcal{K}$.

Quantum noises are driving quantum Langevin equations of all sorts. But it is a remarkable fact that some particular combinations of the quantum noises actually represent well-known classical noises. It can be shown (see S. Attal's lecture in [AJP06]) that:

– the operators $a^0_i(t) + a^i_0(t)$ are naturally isomorphic to the multiplication operators by independent Brownian motions B^i_t acting on their canonical spaces,

– the operators $a^0_i(t) + a^i_0(t) + \lambda a^i_i(t)$ are naturally isomorphic to the multiplication operators by independent compensated Poisson processes X^i_t, with jumps λ and intensity $1/\lambda^2$, acting on their canonical spaces.

In some part of the proof of the main theorem (Theorem III.1) we shall meet, in one particular case, the operator process $(a^i_i(t))_{t\in\mathbb{R}^+}$ alone. Though it is a commutative family of self-adjoint operators and as such they are unitarily equivalent to multiplication operators by a classical process, they are are of deterministic law δ_0 in the reference state of the Fock space (the vacuum state). Hence they bring nothing to the probabilistic interpretation of the associated equation, nor to the associated Lindblad generator. They are of no effect on the small system.

II. 2 Quantum Langevin Equations

Now we recall a few elements of unitary quantum Langevin equations, in the framework of Hudson-Parthasarathy Quantum Stochastic calculus [HP84] [Mey93].

On the space Ψ we consider the following quantum stochastic equation

$$dU_t = \sum_{i,j\in\Lambda\cup\{0\}} L^i_j\, U_t\, da^i_j(t)\,, \tag{II.1}$$

where the a^i_j's are the quantum noises on Φ and where the L^i_j's are operators on $\mathcal{H}$. The following well-known theorem characterizes in terms of the operators L^i_j the fact that the solution (U_t) is made of unitary operators or not.

Theorem II.1 ([HP84]). *If the L^i_j's are bounded operator on $\mathcal{H}$, then Equation (II.1) admits a unique solution on $\mathcal{B}(\mathcal{H}\otimes\Phi)$.*
Furthermore, this solution is made of unitary operators U_t if and only if there exist bounded operators H and S^i_j $(i,j\in\Lambda)$ on $\mathcal{H}$ such that

i) the operator H is selfadjoint,

ii) the operator $\mathbb{S} = \sum_{i,j\in\Lambda} S^i_j \otimes |j\rangle\langle i|$ on $\mathcal{B}(\mathcal{H}\otimes\mathcal{K})$ is unitary

iii) the coefficient L^i_j are of the form:

$$L^0_0 = iH - \frac{1}{2}\sum_{k\in\Lambda}\left(L^0_k\right)^* L^0_k$$
$$L^i_0 = -\sum_{j\in\Lambda}\left(L^0_j\right)^* S^i_j$$
$$L^i_j = S^i_j - \delta_{i,j} I_{\mathcal{H}}\,,$$

for all $i,j\in\Lambda$.

II. 3 Change of noises

Usually the orthonormal basis $(e_i)_{i\in\Lambda}$ is given by the context and one does not change it, once it is fixed. It is clear that the choice of this basis determines the coefficients taking part in Equation (II.1). As an example, consider again the quantum Langevin equation (I.1) given in the introduction, with $S = I_{\mathcal{H}}$:

$$dU_t = \left(iH - \frac{1}{2}L^*L\right)U_t\,dt + L\,U_t\,da^0_1(t) - L^*\,U_t\,da^1_0(t)\,.$$

Here $K = \mathbb{C}e_1$ is one dimensional; one would think that the choice of the basis, i.e. the choice of the unit vector $e_1 \in \mathcal{K}$ is not important. However, suppose that $L^* = \lambda L$ for some $\lambda \in \mathbb{C}$, $|\lambda| = 1$ (this situation may happen whenever the Noise Algebra is commutative, as we shall see in Section III), the previous equation becomes

$$dU_t = \left(iH - \frac{1}{2}L^2\right) U_t\, dt + L\, U_t\, da_1^0(t) - \lambda\, L\, U_t\, da_0^1(t) \tag{II.2}$$

With this choice of a basis it is not clear that the equation is actually driven by a classical noise. However, take $\mu \in \mathbb{C}$ such that $\mu^2 = -\lambda$. If one takes as a new basis the vector $f_1 = \overline{\mu}\, e_1$, then

$$da_1^0(t) = dA^\dagger\left(|e_1\rangle\, 1\!\!1_{[0,t]}\right) = dA^\dagger\left(|\mu\, f_1\rangle\, 1\!\!1_{[0,t]}\right) = \mu\, dA^\dagger\left(|f_1\rangle\, 1\!\!1_{[0,t]}\right) = \mu\, d\tilde{a}_1^0(t)\,.$$

On the other hand, as $da_0^1(t)$ is the adjoint of $da_1^0(t)$, we get

$$da_0^1(t) = \overline{\mu}\, d\tilde{a}_0^1(t)\,.$$

Hence Equation (II.2) becomes

$$dU_t = \left(iH - \frac{1}{2}L^2\right) U_t\, dt + \mu\, L\, U_t\, \left(d\tilde{a}_1^0(t) + d\tilde{a}_0^1(t)\right)\,.$$

We recognize a usual Brownian motion driven quantum Langevin equation

$$dU_t = \left(iH + \frac{1}{2}\tilde{L}^2\right) U_t\, dt + \tilde{L}\, U_t\, dB_t,$$

where $B_t = \tilde{a}_1^0(t) + \tilde{a}_0^1(t)$ is unitarily isomorphic to the multiplication operator by a real Brownian motion and where $\tilde{L}^* = -\tilde{L}$.

It is now clear that in order to unravel classical noises in quantum Langevin equations we must allow changes of basis in $\mathcal{K}$. We shall call such a transformation a *change of noises*.

In order to make the following more readable we fix the following notations. Consider a quantum Langevin equation on Ψ of the form

$$dU_t = L_0^0\, U_t\, dt + \sum_{i=1}^{d} L_i^0\, U_t\, da_i^0(t) + \sum_{i=1}^{d} L_0^i\, U_t\, da_0^i(t) + \sum_{i,j=1}^{d} L_j^i U_t\, da_j^i(t)\,, \tag{II.3}$$

It will be convenient in the sequel to consider the coefficients L_i^0 as a column vector

$$L^0 = \begin{pmatrix} L_1^0 \\ \vdots \\ L_d^0 \end{pmatrix},$$

the coefficients L_0^i as a row vector

$$L_0 = \begin{pmatrix} L_0^1 & \dots & L_0^d \end{pmatrix}$$

and the coefficients L_j^i as a $d \times d$-block-matrix $\mathbb{L}$ such that $\mathbb{L}_{ij} = L_i^j$.

Note that, consistently with these notations, we have

$$\left(L^0\right)^* = \left((L_1^0)^* \quad \dots \quad (L_d^0)^* \right) .$$

Proposition II.1. *Consider a quantum Langevin equation on Ψ of the form*

$$dU_t = L_0^0\, U_t\, dt + \sum_{i=1}^d L_i^0\, U_t\, da_i^0(t) + \sum_{i=1}^d L_0^i\, U_t\, da_0^i(t) + \sum_{i,j=1}^d L_j^i U_t\, da_j^i(t)\,, \tag{II.4}$$

where the quantum noises a_j^i are associated to a given orthonormal basis $(e_i)_{i\in\Lambda}$ of $\mathcal{K}$.

In the orthonormal basis $(f_i)_{i\in\Lambda}$ of $\mathcal{K}$, given by $f_i = We_i$, $i \in \Lambda$, for some unitary operator W on $\mathcal{K}$, Equation (II.5) becomes

$$dU_t = L_0^0\, U_t\, dt + \sum_{i=1}^d \tilde{L}_i^0\, U_t\, d\tilde{a}_i^0(t) + \sum_{i=1}^d \tilde{L}_0^i\, U_t\, d\tilde{a}_0^i(t) + \sum_{i,j=1}^d \tilde{L}_j^i U_t\, d\tilde{a}_j^i(t)\,, \tag{II.5}$$

where

$$\begin{aligned} \tilde{L}^0 &= W^*\, L^0\,, \\ \tilde{L}_0 &= L_0\, W\,, \\ \tilde{\mathbb{L}} &= W^*\, \mathbb{L}\, W\,. \end{aligned}$$

Proof. We have $e_i = \sum_{j\in\Lambda} (W^*)_{ji}\, f_j$, so that

$$\begin{aligned} da_i^0(t) &= \sum_{j\in\Lambda} (W^*)_{ji}\, d\tilde{a}_j^0(t) \\ da_0^i(t) &= \sum_{j\in\Lambda} W_{ij}\, d\tilde{a}_0^j(t) \\ da_j^i(t) &= \sum_{k,l\in\Lambda} W_{ik}\, (W^*)_{lj}\, d\tilde{a}_l^k(t)\,. \end{aligned}$$

This gives

$$\begin{aligned} dU_t = L_0^0\, U_t\, dt + \sum_{j=1}^d \sum_{i=1}^d (W^*)_{ji}\, L_i^0\, U_t\, d\tilde{a}_j^0(t) + \sum_{j=1}^d \sum_{i=1}^d L_0^i W_{ij}\, U_t\, d\tilde{a}_0^i(t) + \\ + \sum_{k,l=1}^d \sum_{i,j=1}^d (W^*)_{lj}\, \mathbb{L}_{ji}\, W_{ik}\, U_t\, d\tilde{a}_l^k(t)\,. \end{aligned}$$

This gives the result. □

Regarding the case of unitary-valued quantum Langevin equations, the proposition above shows that the conditions on the L_j^i's are not affected by changes of noise, as is summarized below.

Proposition II.2. *Consider a unitary-valued quantum Langevin equation of the form*

$$dU_t = -\left(i\,H + \frac{1}{2}\sum_{k\in\Lambda}\left(L_k^0\right)^*\,L_k^0\right)U_t\,dt + \sum_{k\in\Lambda} L_k^0\,U_t\,da_k^0(t)$$
$$+\sum_{k\in\Lambda}\left(-\sum_{l\in\Lambda}\left(L_l^0\right)^*\,S_l^k\right)U_t\,da_0^k(t) + \sum_{k,l\in\Lambda}\left(S_l^k - \delta_{k,l}\,I_{\mathcal{H}}\right)U_t\,da_l^k(t)\,, \quad \text{(II.6)}$$

where H is selfadjoint and the operator $\mathbb{S} = \sum_{i,j\in\Lambda} S_j^i \otimes |j\rangle\langle i|$ on $\mathcal{B}(\mathcal{H}\otimes\mathcal{K})$ is unitary. Then, after a change of noise of the form $f_i = W e_i$, $i = 1,\dots,d$, the equation becomes

$$dU_t = -\left(i\,H + \frac{1}{2}\sum_{k\in\Lambda}\left(\tilde{L}_k^0\right)^*\,\tilde{L}_k^0\right)U_t\,dt + \sum_{k\in\Lambda} \tilde{L}_k^0\,U_t\,d\tilde{a}_k^0(t)$$
$$+\sum_{k\in\Lambda}\left(-\sum_{l\in\Lambda}\left(\tilde{L}_l^0\right)^*\,\tilde{S}_l^k\right)U_t\,d\tilde{a}_0^k(t) + \sum_{k,l\in\Lambda}\left(\tilde{S}_l^k - \delta_{k,l}\,I_{\mathcal{H}}\right)U_t\,d\tilde{a}_l^k(t)\,, \quad \text{(II.7)}$$

where

$$\tilde{L}^0 = W^*\,L^0$$
$$\tilde{S} = W^*\,S\,W\,.$$

II. 4 The Noise Algebra

In a previous article on one-step evolutions [Bar15], I. Bardet defined the Environment Algebra as the Von Neumann subalgebra of the environment generated by the unitary operator of a one-step evolution on the bipartite system $\mathcal{H}\otimes\mathcal{K}$. We recall here the basic definitions and the main result on the decomposition of the environment between a commutative and a quantum part.

Let $\mathbb{S}$ be a unitary operator on $\mathcal{H}\otimes\mathcal{K}$. For $f,g\in\mathcal{H}$, we define:

$$\mathbb{S}(f,g) = \mathrm{Tr}_{|g\rangle\langle f|}[\mathbb{S}]\,, \qquad \mathbb{S}^*(f,g) = \mathrm{Tr}_{|g\rangle\langle f|}[\mathbb{S}^*]\,. \quad \text{(II.8)}$$

Those operators can be thought of as pictures of $\mathbb{S}$ taken from $\mathcal{K}$ but with different angles.

Definition II.1. *Let $\mathbb{S}$ be a unitary operator on $\mathcal{H}\otimes\mathcal{K}$. We call* Environment Algebra *the von Neumann algebra $\mathcal{A}(\mathbb{S})$ generated by the $\mathbb{S}(f,g)$, that is*

$$\mathcal{A}(\mathbb{S}) = \{\mathbb{S}(f,g),\ \mathbb{S}^*(f,g);\quad f,g\in\mathcal{H}\}''\,. \quad \text{(II.9)}$$

The point with this definition is that it fits with the following characterization.

Proposition II.3. *Let $\mathbb{S}$ be a unitary operator on $\mathcal{H}\otimes\mathcal{K}$. Then $\mathcal{A}(\mathbb{S})$ is the smallest von Neumann subalgebra of $\mathcal{B}(\mathcal{K})$ such that $\mathbb{S}$ and $\mathbb{S}^*$ belong to $\mathcal{B}(\mathcal{H})\otimes\mathcal{A}(\mathbb{S})$, i.e. if $\mathcal{A}$*

is another von Neumann algebra such that $\mathbb{S}$ *and* $\mathbb{S}^*$ *belong to* $\mathcal{B}(\mathcal{H}) \otimes \mathcal{A}$*, then* $\mathcal{A}(\mathbb{S}) \subset \mathcal{A}$*. Furthermore, its commutant is given by*

$$\mathcal{A}(\mathbb{S})' = \{Y \in \mathcal{B}(\mathcal{K}), \quad [I_{\mathcal{H}} \otimes Y\,,\,\mathbb{S}] = [I_{\mathcal{H}} \otimes Y\,,\,\mathbb{S}^*] = 0\}\,. \tag{II.10}$$

We now give the definition of the commutative part of the environment and the associated decomposition.

Definition II.2. *Let* $\mathbb{S}$ *be a unitary operator on* $\mathcal{H} \otimes \mathcal{K}$ *and let* $\tilde{\mathcal{K}}$ *be a subspace of* $\mathcal{K}$*. We say that* $\tilde{\mathcal{K}}$ *is a* Commutative Subspace of the Environment *if* $\tilde{\mathcal{K}} \neq \{0\}$ *and:*

i) $\mathcal{H} \otimes \tilde{\mathcal{K}}$ *and* $\mathcal{H} \otimes \tilde{\mathcal{K}}^{\perp}$ *are stable by* $\mathbb{S}$*,*

ii) $\mathcal{A}(\tilde{\mathbb{S}})$ *is commutative, where* $\tilde{\mathbb{S}}$ *is the restriction of* $\mathbb{S}$ *to* $\mathcal{H} \otimes \tilde{\mathcal{K}}$*.*

We then have the following Decomposition Theorem, which is proved in [Bar15].

Theorem II.2. *The environment Hilbert space* $\mathcal{K}$ *is the orthogonal direct sum of two subspaces* $\mathcal{K}_c$ *and* $\mathcal{K}_q$*, such that either* $\mathcal{K}_c = \{0\}$ *or*

i) $\mathcal{K}_c$ *is a commutative subspace of the environment.*

ii) If $\tilde{\mathcal{K}}$ *is any commutative subspace of the environment, then* $\tilde{\mathcal{K}} \subset \mathcal{K}_c$*.*

iii) The restriction of $\mathbb{S}$ *to* $\mathcal{H} \otimes \mathcal{K}_q$ *does not have any commutative subspace.*

We now come back to our continuous time scenario. In this situation, we define the *Noise Algebra* as an algebra which encodes the structure of the noise in the unitary quantum Langevin equation.

Definition II.3. *Let* (U_t) *be the unitary-valued solution of a quantum Langevin Equation. The* Noise Algebra *(at time* t*) is defined by*

$$\mathcal{A}_t(U) = \left\{\; Tr_{|f\rangle\langle g|}\,[U_s]\,,\;\; Tr_{|f\rangle\langle g|}\,[U_s^*]\,; \quad f, g \in \mathcal{H}, \quad 0 < s \le t\right\}''\,. \tag{II.11}$$

It is obvious that it is enough to consider only the vectors of an orthonormal basis of $\mathcal{H}$ in this definition. Let $(g_i)_{i \in I}$ be such a basis. For simplicity we adopt the following notation: if T is a bounded operator on $\mathcal{H} \otimes \Phi$, we write T^{ij} for $\mathrm{Tr}_{|f_i\rangle\langle f_j|}\,[T]$, $i, j \in I$.

III The case of a Commutative Environment

The aim of this section is to completely characterize those unitary quantum Langevin equations for which $\mathcal{A}_t(U)$ is commutative. We do that in subsection III. 1. The characterization is first algebraic, then interpreted in terms of classical noises. Finally, we apply this characterization to give the general form of the associated Lindblad generator in subsection III. 2.

III. 1 Characterization of Commutative Noise Algebras

We shall need the following notations. If the matrix $\mathbb{S}$, as a block-matrix on $\mathcal{K}$, is diagonalizable in some orthonormal basis of $\mathcal{K}$, we put $\mathcal{K}_{\mathbb{W}}$ to be the maximal subspace of $\mathcal{K}$ such that $\mathbb{S}$ acts as the identity operator on $\mathcal{H}\otimes\mathcal{K}_{\mathbb{W}}$ and we put $\mathcal{K}_{\mathbb{P}} = \mathcal{K}_{\mathbb{W}}^{\perp}$. We consider some orthonormal basis $\{f_1, \dots, f_m, f_{m+1}, \dots, f_d\}$ of $\mathcal{K}$ adapted to the decomposition $\mathcal{K} = \mathcal{K}_{\mathbb{W}} \oplus \mathcal{K}_{\mathbb{P}}$, where m is the dimension of $\mathcal{K}_{\mathbb{W}}$. In this basis, the matrix $\mathbb{S}$ can then be written as

$$\mathbb{S} = \begin{pmatrix} I_{\mathcal{H}\otimes\mathcal{K}_{\mathbb{W}}} & 0 \\ 0 & \mathbb{S}_{\mathbb{P}} \end{pmatrix}, \quad \mathbb{S}_{\mathbb{P}} = \begin{pmatrix} S_1 & 0 & \cdots & 0 \\ 0 & S_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & S_{d-m} \end{pmatrix}.$$

We shall denote by $\Lambda_{\mathbb{W}}$ the set of indices $\{1, \dots, m\}$ and by $\Lambda_{\mathbb{P}}$ the other ones.

We can now state the first main result of this article, which completely characterizes the commutativity of the algebra $\mathcal{A}_t(U)$ in terms of algebraic properties of the coefficients L^i_j.

Theorem III.1. *Consider a unitary-valued quantum Langevin equation of the form*

$$\begin{aligned} dU_t = \left(i\, H - \frac{1}{2} \sum_{k\in\Lambda} (L^0_k)^*\, L^0_k \right) U_t\, dt + \sum_{k\in\Lambda} L^0_k\, U_t\, da^0_k(t) \\ + \sum_{k\in\Lambda} \left(-\sum_{l\in\Lambda} (L^0_l)^*\, S^k_l \right) U_t\, da^k_0(t) + \sum_{k,l\in\Lambda} \left(S^k_l - \delta_{k,l}\, I_{\mathcal{H}} \right) U_t\, da^k_l(t)\,, \end{aligned} \tag{III.1}$$

where H is selfadjoint and the operator $\mathbb{S} = \sum_{i,j\in\Lambda} S^i_j \otimes |j\rangle\langle i|$ on $\mathcal{B}(\mathcal{H}\otimes\mathcal{K})$ is unitary. Then the following assertions are equivalent.

1) The algebra $\mathcal{A}_t(U)$ is commutative for all $t > 0$.

2) The algebra $\mathcal{A}_t(U)$ is commutative for some $t > 0$.

3) The matrix $\mathbb{S}$, as a block-matrix on $\mathcal{K}$, is diagonalizable in some orthonormal basis of $\mathcal{K}$ and, considering the coefficients L^i_j after the appropriate change of noise, we have that

i) there exists a symmetric unitary operator W on $\mathcal{K}_{\mathbb{W}}$ such that

$$\begin{pmatrix} (L^0_1)^* \\ \vdots \\ (L^0_m)^* \end{pmatrix} = W \begin{pmatrix} L^0_1 \\ \vdots \\ L^0_m \end{pmatrix}, \tag{III.2}$$

ii) for all $i \in \Lambda_{\mathbb{P}}$, there exists $\lambda_i \in \mathbb{C}$ such that

$$L^0_i = \lambda_i (S_i - I_{\mathcal{H}})\,. \tag{III.3}$$

4) There exists a change of noise such that Equation (III.1) is of the form

$$dU_t = A_0\, U_t\, dt + \sum_{i\in\Lambda_{\mathbb{W}}} A_i\, U_t\, dW_t^i + \sum_{i\in\Lambda_{\mathbb{P}}} B_i\, U_t\, dX_t^i\,, \tag{III.4}$$

where A_0, $\{A_i,\, i\in\Lambda_{\mathbb{W}}\}$ and $\{B_i,\, i\in\Lambda_{\mathbb{P}}\}$ are bounded operators on $\mathcal{H}$, where W^i, $i\in\Lambda_{\mathbb{W}}$, are (multiplication operators by) standard Brownian motions, where X^i, $i\in\Lambda_{\mathbb{P}}$, are (multiplication operators by) compensated Poisson processes and such that all the processes W^i and X^j are independent.

Proof. Obviously 1) implies 2).

Proof of 2) implies 3) :

Let us write Equation (III.1) as

$$dU_t = \sum_{i,j\in\Lambda\cup\{0\}} L_j^i\, U_t\, da_j^i(t)\,,$$

for short.

We consider a fixed orthonormal basis $(g_i)_{i\in I}$ of $\mathcal{H}$ and the corresponding notation T^{ij} for bounded operators on $\mathcal{H}\otimes\Phi$. We first exploit the relation $[U_s^{kl}\,,\, U_s^{mn}]=0$ for all $s<t$, all $k,l,m,n\in I$. Differentiating this equality and using the Itô rule we get

$$\begin{aligned}0 = &\sum_{i,j\in\Lambda\cup\{0\}} \left[\left(L_j^i U_s\right)^{kl} U_s^{mn} + U_s^{kl}\left(L_j^i U_s\right)^{mn} + \sum_{i'\in\Lambda}\left(L_j^{i'} U_s\right)^{kl}\left(L_{i'}^i U_s\right)^{mn}\right] da_j^i(s)\\ &- \sum_{i,j\in\Lambda\cup\{0\}} \left[\left(L_j^i U_s\right)^{mn} U_s^{kl} + U_s^{mn}\left(L_j^i U_s\right)^{kl} + \sum_{i'\in\Lambda}\left(L_j^{i'} U_s\right)^{mn}\left(L_{i'}^i U_s\right)^{kl}\right] da_j^i(s)\,.\end{aligned} \tag{III.5}$$

Identifying each of the coefficients of $da_j^i(s)$ to 0 and taking the limit $s\to 0$, we get, for all $k,l,m,n\in I$, all $i,j\in\Lambda\cup\{0\}$,

$$\left(L_j^i\right)^{kl} + \left(L_j^i\right)^{mn} + \sum_{i'\in\Lambda}\left(L_j^{i'}\right)^{kl}\left(L_{i'}^i\right)^{mn} - \left(L_j^i\right)^{mn} - \left(L_j^i\right)^{kl} - \sum_{i'\in\Lambda}\left(L_j^{i'}\right)^{mn}\left(L_{i'}^i\right)^{kl} = 0\,,$$

that is,

$$\sum_{i'\in\Lambda}\left(L_j^{i'}\right)^{kl}\left(L_{i'}^i\right)^{mn} - \left(L_j^{i'}\right)^{mn}\left(L_{i'}^i\right)^{kl} = 0\,. \tag{III.6}$$

Consider the block-matrix $\mathbb{L}$ given by $\mathbb{L}_{ij}=L_i^j$, $i,j\in\Lambda$. We denote by $\mathbb{L}^{kl}$ the matrix with coefficients $(\mathbb{L}^{kl})_{ij}=(\mathbb{L}_{ij})^{kl}=(L_i^j)^{kl}$. With these notations we have

$$\left(\mathbb{L}^{kl}\mathbb{L}^{mn}\right)_{ij} = \sum_{i'\in\Lambda}\left(\mathbb{L}^{kl}\right)_{ii'}\left(\mathbb{L}^{mn}\right)_{i'j} = \sum_{i'\in\Lambda}\left(L_i^{i'}\right)^{kl}\left(L_{i'}^j\right)^{mn}\,.$$

This way, Equation (III.6) means

$$\mathbb{L}^{kl}\mathbb{L}^{mn} - \mathbb{L}^{mn}\mathbb{L}^{kl} = 0\,. \tag{III.7}$$

We now exploit the relation $[(U_s^*)^{kl}, U_s^{mn}] = 0$ for all $s < t$, all $k,l,m,n \in I$. Differentiating this equality and using the Itô rule we get

$$0 = \sum_{i,j\in\Lambda\cup\{0\}} \left[\left(U_s^*(L_i^j)^*\right)^{kl} U_s^{mn} + (U_s^*)^{kl} \left(L_j^i U_s\right)^{mn} + \sum_{i'\in\Lambda} \left(U_s^*(L_{i'}^j)^*\right)^{kl} \left(L_{i'}^i U_s\right)^{mn} \right] da_j^i(s)$$
$$- \sum_{i,j\in\Lambda\cup\{0\}} \left[\left(L_j^i U_s\right)^{mn} (U_s^*)^{kl} + U_s^{mn} \left(U_s^*(L_i^j)^*\right)^{kl} + \sum_{i'\in\Lambda} \left(L_j^{i'} U_s\right)^{mn} \left(U_s^*(L_i^{i'})^*\right)^{kl} \right] da_j^i(s)\,.$$

Identifying each of the coefficients of $da_j^i(s)$ to 0 and taking the limit $s \to 0$, we get, for all $k,l,m,n \in I$, all $i,j \in \Lambda\cup\{0\}$,

$$\left((L_i^j)^*\right)^{kl} + \left(L_j^i\right)^{mn} + \sum_{i'\in\Lambda} \left((L_{i'}^j)^*\right)^{kl} \left(L_{i'}^i\right)^{mn} - \left(L_j^i\right)^{mn} - \left((L_i^j)^*\right)^{kl} - \sum_{i'\in\Lambda} \left(L_j^{i'}\right)^{mn} \left((L_i^{i'})^*\right)^{kl} = 0\,,$$

that is,

$$\sum_{i'\in\Lambda} \left((L_{i'}^j)^*\right)^{kl} \left(L_{i'}^i\right)^{mn} - \left(L_j^{i'}\right)^{mn} \left((L_i^{i'})^*\right)^{kl} = 0\,. \tag{III.8}$$

The block-matrix $\mathbb{L}$ defined above satisfies $(\mathbb{L}^*)_{ij} = (\mathbb{L}_{ji})^* = (L_j^i)^*$, so that

$$\left((\mathbb{L}^*)^{kl}\right)_{ij} = ((\mathbb{L}^*)_{ij})^{kl} = \left((L_i^j)^*\right)^{kl}\,.$$

This way, Equation (III.8) means

$$(\mathbb{L}^*)^{kl}\mathbb{L}^{mn} - \mathbb{L}^{mn}(\mathbb{L}^*)^{kl} = 0\,. \tag{III.9}$$

With Equations (III.7) and (III.9) we have proved that all the matrices $\mathbb{L}^{kl}$ and $(\mathbb{L}^*)^{mn}$ commute. As $\mathbb{S} = \mathbb{L} + I_{\mathcal{H}\otimes\mathcal{K}}$, we get that all the matrices $\mathbb{S}^{kl}$ and $(\mathbb{S}^*)^{mn}$ commute too, so that the algebra $\mathcal{A}(\mathbb{S})$ is commutative.

As a consequence the block-matrix $\mathbb{S}$ can be block-diagonalized. We can write, as announced in the theorem

$$\mathbb{S} = \begin{pmatrix} I_{\mathcal{H}\otimes\mathcal{K}_{\mathbb{W}}} & 0 \\ 0 & \mathbb{S}_{\mathbb{P}} \end{pmatrix}$$

We now make a change of noise adapted to the decomposition of $\mathcal{K}$ as the direct sum of $\mathcal{K}_{\mathbb{W}}$ and $\mathcal{K}_{\mathbb{P}}$. In particular, in the new noises, we have

$$L_j^i = 0 \text{ for all } i,j \in \Lambda,\ i \neq j\,.$$

Following our notations above, we put $S_i = S_i^i$ for all i. Note that the coefficients S_i have to be unitary operators on $\mathcal{H}$, for $\mathbb{S}$ to be unitary on $\mathcal{H}\otimes\mathcal{K}$.

With these reductions, Equation (III.6) becomes, when $i = 0$ and $j \neq 0$

$$(S_i - I)^{kl} \left(L_j^0\right)^{mn} = (S_i - I)^{mn} \left(L_j^0\right)^{kl}\,. \tag{III.10}$$

On the other hand, Equation (III.8) reduces, when $i = j = 0$, to

$$\sum_{i\in\Lambda} \left((L_i^0)^*\right)^{kl} \left(L_i^0\right)^{mn} = \sum_{i\in\Lambda} \left(L_0^i\right)^{mn} \left((L_0^i)^*\right)^{kl} . \tag{III.11}$$

Let us consider some index $i \in \Lambda_{\mathbb{P}}$, that is, for which $S_i \neq I$. In particular there exist $k, l \in I$ such that $(S_i - I_{\mathcal{H}})^{kl} \neq 0$. Equation (III.10) then gives for all $m, n \in I$,

$$\left(L_i^0\right)^{mn} = \frac{(L_i^0)^{kl}}{(S_i - I)^{kl}} (S_i - I)^{mn} .$$

Defining $\lambda_i = (L_i^0)^{kl} / (S_i - I)^{kl}$, this gives $L_i^0 = \lambda_i (S_i - I)$. This proves the property (III.3).

We now come back to Equation (III.11), separating the indices in $\Lambda_{\mathbb{P}}$ and those in $\Lambda_{\mathbb{W}}$. Using the fact that $L_0^i = -(L_i^0)^*$ when i belongs to $\Lambda_{\mathbb{W}}$ and the relation $L_i^0 = \lambda_i(S_i - I)$ when i belongs to $\Lambda_{\mathbb{P}}$, we get

$$\begin{aligned}\sum_{i\in\Lambda_{\mathbb{W}}} \left((L_i^0)^*\right)^{kl} \left(L_i^0\right)^{mn} + \sum_{i\in\Lambda_{\mathbb{P}}} |\lambda_i|^2 \left((S_i - I)^*\right)^{kl} (S_i - I)^{mn} = \\ = \sum_{i\in\Lambda_{\mathbb{W}}} \left((L_i^0)^*\right)^{mn} \left(L_i^0\right)^{kl} + \sum_{i\in\Lambda_{\mathbb{P}}} |\lambda_i|^2 \left((S_i - I)^*\right)^{kl} (S_i - I)^{mn} .\end{aligned}$$

This reduces to

$$\sum_{i\in\Lambda_{\mathbb{W}}} \left((L_i^0)^*\right)^{kl} \left(L_i^0\right)^{mn} = \sum_{i\in\Lambda_{\mathbb{W}}} \left((L_i^0)^*\right)^{mn} \left(L_i^0\right)^{kl} ,$$

or else

$$\sum_{i\in\Lambda_{\mathbb{W}}} \overline{(L_i^0)^{lk}} \left(L_i^0\right)^{mn} = \sum_{i\in\Lambda_{\mathbb{W}}} \overline{(L_i^0)^{nm}} \left(L_i^0\right)^{kl} . \tag{III.12}$$

Put $u(k,l) = ((L_i^0)^{kl})_{i\in\Lambda_{\mathbb{W}}} \in \mathbb{C}^m$ and $v(k,l) = (\overline{(L_i^0)^{lk}})_{i\in\Lambda_{\mathbb{W}}} \in \mathbb{C}^m$, for all $k, l \in I$. Equation (III.12) then becomes

$$\langle u(l,k)\,,\, u(m,n)\rangle = \langle v(l,k)\,,\, v(m,n)\rangle ,$$

for all $k, l, m, n \in I$.

We claim that this implies that there exists a unitary operator W on $\mathbb{C}^m$ such that $W\,u(k,l) = v(k,l)$ for all $k, l \in I$. Indeed, we can assume that the family $\{u(k,l)\,;\ k, l \in I\}$ has maximal rank in $\mathbb{C}^m$, otherwise we complete it. The family $\{v(k,l)\,;\ k, l \in I\}$ has same rank, so we complete it in the same way. Consider the matrices $U = (u(k,l)) \in \mathcal{M}_{m^2,m}(\mathbb{C})$ and $V = (v(k,l)) \in \mathcal{M}_{m^2,m}(\mathbb{C})$. By hypothesis we have $V^*V = U^*U$. Consider the polar decomposition of U and V:

$$U = M\sqrt{U^*U} \qquad \text{and} \qquad V = N\sqrt{V^*V} ,$$

where $M, N : \mathbb{C}^m \to C^{m^2}$ are partial isometries, where $\mathrm{Ker}\, M = \mathrm{Ker}\, U$ and $\mathrm{Ker}\, N = \mathrm{Ker}\, V$, where $\mathrm{Im}\, M = \mathrm{Im}\, U = \mathbb{C}^m = \mathrm{Im}\, V = \mathrm{Im}\, N$. Put W_1 to be a unitary operator on $\mathbb{C}^d$ which agrees with M on $(\mathrm{Ker}\ M)^\perp$ and W_2 to be a unitary operator on $\mathbb{C}^m$ which agrees with N on $(\mathrm{Ker}\ N)^\perp$. Putting $W = W_2 W_1^*$ it is easy to check that $V = WU$. This proves the claim.

Now, let us prove that W has also to be symmetric, that is $W^t = W$. We have proved the relation $V = WU$, that is for all $k, l \in \Lambda$

$$\begin{pmatrix} ((L_1^0)^*)^{kl} \\ \vdots \\ ((L_m^0)^*)^{kl} \end{pmatrix} = W \begin{pmatrix} (L_1^0)^{kl} \\ \vdots \\ (L_m^0)^{kl} \end{pmatrix},$$

for all $k, l \in \Lambda$. Hence we have

$$\begin{pmatrix} (L_1^0)^* \\ \vdots \\ (L_m^0)^* \end{pmatrix} = W \begin{pmatrix} L_1^0 \\ \vdots \\ L_m^0 \end{pmatrix},$$

so that

$$\begin{pmatrix} L_1^0 & \dots & L_m^0 \end{pmatrix} = \begin{pmatrix} (L_1^0)^* & \dots & (L_m^0)^* \end{pmatrix} W^*$$

and finally

$$\begin{pmatrix} L_1^0 \\ \vdots \\ L_m^0 \end{pmatrix} = \overline{W} \begin{pmatrix} (L_1^0)^* \\ \vdots \\ (L_m^0)^* \end{pmatrix}.$$

We have proved that $\overline{W} = W^*$, hence $W^t = W$. We have proved the property (III.2). We have proved that 1) implies 3).

Proof of 3) implies 4) :

If the coefficients L_j^i satisfy all the properties described in 3), then, after the adequate change of noise, Equation (III.1) reduces to

$$\begin{aligned} dU_t = K_0\, U_t\, dt + \sum_{k \in \Lambda_{\mathbb{W}}} L_k^0\, U_t\, da_k^0(t) + \sum_{k \in \Lambda_{\mathbb{W}}} L_0^k\, U_t\, da_0^k(t) + \\ + \sum_{k \in \Lambda_{\mathbb{P}}} \lambda_k\, (S_k - I)\, U_t\, da_k^0(t) + \sum_{k \in \Lambda_{\mathbb{P}}} \overline{\lambda_k}\, (S_k - I)\, U_t\, da_0^k(t) + \sum_{k \in \Lambda_{\mathbb{P}}} (S_k - I)\, U_t\, da_k^k(t)\,, \end{aligned} \tag{III.13}$$

where we do not need to detail the operator K_0 anymore here. We first focus on the Wiener part. Write L_0 the column of the coefficients L_i^0, $i \in \Lambda_{\mathbb{W}}$ and L^0 the row of the coefficients L_0^i, $i \in \Lambda_{\mathbb{W}}$. Recall that $L_0 = -(L^0)^*$, but we also have as a condition in 3) that $((L^0)^*)^t = W(L^0)^*$, or else $(L^0)^* = (L^0)^t\, W^t$.

The matrix W is a symmetric unitary operator. By the Takagi Factorization theorem (see [HJ12] for example), every symmetric complex square matrix can be decomposed as

$V^t\,D\,V$, where V is unitary and D is diagonal with real positive entries. Thus W admits such a decomposition. But the fact that W is unitary gives

$$I = W^*W = V^*\,D\,\overline{V}\,V^t\,D\,V = V^*\,D^2\,V\,.$$

In particular $D^2 = I$ and thus $D = I$. We have proved that W is the form $V^t\,V$ for some unitary V. Actually, we apply this decomposition to $-W$ instead and we write $-W = V^t\,V$ for some unitary matrix V.

As we said above, we have $L_0 = -(L^0)^t\,W^t$, which gives

$$L_0 = (L^0)^t\,V^t\,V = (V\,L^0)^t\,V\,.$$

On the other hand we have $L^0 = V^*\,(V\,L^0)$. Hence if we put $K = V\,L^0$, the part

$$\sum_{k\in\Lambda_{\mathbb{W}}} L^0_k\,U_t\,da^0_k(t) + \sum_{k\in\Lambda_{\mathbb{W}}} L^k_0\,U_t\,da^k_0(t)$$

of Equation (III.13) can now be written as

$$\sum_{k\in\Lambda_{\mathbb{W}}} (V^*K)_k\,U_t\,da^0_k(t) + \sum_{k\in\Lambda_{\mathbb{W}}} ((K^t)V)_k\,U_t\,da^k_0(t)\,.$$

Hence, by Proposition II.1, if we apply the change of noise associated to V to the orthonormal basis of $\mathcal{K}_{\mathbb{W}}$, we obtain a term of the form

$$\sum_{k\in\Lambda_{\mathbb{W}}} K_k\,U_t\,da^0_k(t) + \sum_{k\in\Lambda_{\mathbb{W}}} K_k\,U_t\,da^k_0(t) = \sum_{k\in\Lambda_{\mathbb{W}}} K_k\,U_t\,dW^k_t\,.$$

Let us now concentrate on the part indexed by $\Lambda_{\mathbb{P}}$ in Equation (III.13). If we decompose each λ_k into $\rho_k\,e^{i\theta_k}$, for those $\lambda_k \neq 0$, then this part of (III.13) can be written as

$$\sum_{k\in\Lambda_{\mathbb{P}}} \rho_k\,(S_k - I)\,U_t\left(e^{i\theta_k}\,da^0_k(t) + e^{-i\theta_k}\,da^k_0(t) + \frac{1}{\rho_k}\,da^k_k(t)\right).$$

After a change of noise $f_k \mapsto e^{i\theta_k}\,f_k$, the expression above reduces to

$$\sum_{k\in\Lambda_{\mathbb{P}}} \rho_k\,(S_k - I)\,U_t\left(da^0_k(t) + da^k_0(t) + \frac{1}{\rho_k}\,da^k_k(t)\right),$$

which is exactly of the form announced in 4), that is,

$$\sum_{k\in\Lambda_{\mathbb{P}}} B_k\,U_t\,dX^k_t\,.$$

For those $\lambda_k = 0$, as we discussed in Subsection II. 1, the equation gives rise to a term of the form

$$(S_k - I)\,U_t\,da^k_k(t)$$

which is of no contribution in the probabilistic interpretation of the equation, nor the effect of the evolution on the small system $\mathcal{H}$.

Proof of 4) implies 1) : We start with a quantum Langevin equation of the form

$$dU_t = A_0\, U_t\, dt + \sum_{i\in\Lambda_{\mathbb{W}}} A_i\, U_t\, dW_t^i + \sum_{i\in\Lambda_{\mathbb{P}}} B_i\, U_t\, dX_t^i\,, \tag{III.14}$$

as stated in 4).

Consider the von Neumann algebra $\mathcal{N}_t$ generated by the operators $\{W_s^i\,,\, X_s^i\,;\; i \in \Lambda\,,\, s \le t\}$. It is clear that it is a commutative von Neumann algebra, for they are all self-adjoint and pairwise commuting operators. We claim that $\mathcal{A}_t(U) \subset \mathcal{N}_t$, let us prove this fact.

Indeed, applying the $\mathrm{Tr}_{|e_k\rangle\langle e_l|}$ to Equation (III.4) gives:

$$U_t^{kl} = I + \sum_{p\in I}\int_0^t A_0^{kp}\, U_s^{pl}\, ds + \sum_{i\in\Lambda_{\mathbb{W}}}\sum_{p\in I}\int_0^t A_i^{kp}\, U_s^{pl}\, dW_s^i + \sum_{i\in\Lambda_{\mathbb{P}}}\sum_{p\in I}\int_0^t B_i^{kp}\, U_s^{pl}\, dX_s^i\,.$$

It is a system of integral equations in the U^{kl} variables. With the conditions on the coefficients of Equation (III.4), it is rather standard to prove that the solutions U_t^{kl} can be obtained as the strong limit of a Picard iteration:

$$\begin{aligned}
\left(U_s^{kl}\right)^0 &= I\,,\ \text{for all } s \le t\,,\ \text{for all } k, l \in I,\\
\left(U_t^{kl}\right)^{n+1} &= I + \sum_{p\in I}\int_0^t A_0^{kp}\, \left(U_s^{pl}\right)^n\, ds + \sum_{i\in\Lambda_{\mathbb{W}}}\sum_{p\in I}\int_0^t A_i^{kp}\, \left(U_s^{pl}\right)^n\, dW_s^i +\\
&\quad + \sum_{i\in\Lambda_{\mathbb{P}}}\sum_{p\in I}\int_0^t B_i^{kp}\, \left(U^p l_s\right)^n\, dX_s^i\,.
\end{aligned}$$

All the U_s^{kl}, $s \le t$, $k, l \in I$, belong to $\mathcal{N}_t$. By induction, if all the $(U_s^{kl})^n$, $s \le t$, $k, l \in I$, belong to $\mathcal{N}_t$, then so do all the $(U_s^{kl})^{n+1}$, $s \le t$, $k, l \in I$, for in the equation above $(U_s^{kl})^{n+1}$ is obtained as strong limit of Riemann sums of elements of $\mathcal{N}_t$. Hence, passing to the strong limit, all the U_s^{kl}, $s \le t$, $k, l \in I$, belong to $\mathcal{N}_t$. The same argument works also for the $(U_s^*)^{kl}$. As $\mathcal{A}_t(U)$ is the von Neumann algebra generated by the U_s^{kl} and the $(U_s^*)^{kl}$, we get the announced inclusion: $\mathcal{A}_t(U) \subset \mathcal{N}_t$.

As a consequence, $\mathcal{A}_t(U)$ is commutative, and this holds for all $t \in \mathbb{R}^+$. This proves 1).

Strictly speaking, Equation (III.14) may contain additional terms with $da_k^k(t)$ alone as a driving noise. These terms corresponding to the cases $\lambda_k = 0$ in 3). But these terms change nothing on the proof of "4) implies 1)", as we can add them to the commutative algebra $\mathcal{N}_t$ and carry on with exactly the same proof.

The theorem is proved. □

We thought it could be of interest to make explicit the same theorem as above, but in the case $d = 1$, for in this case it takes a particularly simple form.

Theorem III.2. *Consider the unitary solution (U_t) of a quantum Langevin equation:*

$$dU_t = \left(iH + \frac{1}{2}L^* L\right) U_t\, dt + L\, U_t\, da_1^0(t) - L^*\, S\, U_t da_0^1(t) + (S - I_{\mathcal{H}})\, U_t\, da_1^1(t),$$

with $H, L, S \in \mathcal{B}(\mathcal{H})$, $H = H^*$ and S a unitary operator. Then the following assertions are equivalent:

1) $\mathcal{A}_t(U)$ is commutative for some $t > 0$.

2) $\mathcal{A}_t(U)$ is commutative for all $t > 0$.

3) One of the following two conditions holds:
– either $S = I_{\mathcal{H}}$ and there exists $\theta \in \mathbb{R}$ such that $L^* = e^{i\theta} L$;
– or there exists a complex number λ such that $L = \lambda(S - I_{\mathcal{H}})$.

4) After the appropriate change of noise Equation (I.1) takes the form of either Equation (I.2) or Equation (I.3) with a Poisson process of intensity $|\lambda|$ depending whether the first or the second case holds in 3).

III. 2 Lindblad Generators

One of the main motivation in constructing the unitary solution of quantum Langevin Equations is that their solutions gives a cocycle unitary dilation of Quantum Markov Semigroups (QMS). Indeed, let $U.$ be the unique unitary solution of Equation (III.1). Using the Itô table for the quantum noises one shows that if we put

$$\mathcal{P}_t(X) = \langle \Omega, U_t^* (X \otimes I_\Phi) U_t \Omega \rangle \qquad X \in \mathcal{B}(\mathcal{H}), \tag{III.15}$$

for all $t \in \mathbb{R}^+$, then this defines a norm-continuous Quantum Markov Semigroup on $\mathcal{B}(\mathcal{H})$. Moreover, its Lindblad generator $\mathcal{L}(\cdot)$ is given by

$$\mathcal{L}(x) = -i[H, X] + \frac{1}{2} \sum_{k \in \Lambda} (L_k^* L_k X + X L_k^* L_k + 2 L_k^* X L_k)\,. \tag{III.16}$$

We see that the unitary operator $\mathbb{S}$ does not play any role in this generator. For this reason, it is called the *gauge* of the quantum Langevin Equation.

Any generator of a norm-continuous QMS on $\mathcal{B}(\mathcal{H})$ can be written under the form (III.16), so that the QMS admits a cocycle unitary dilation $U.$ solution of a quantum Langevin equation.

We now illustrate Theorem III.1 with two applications to QMS: essentially commutative dilation and detailed balance condition.

A result of Kummerer and Maassen [KM87] characterizes the particular structure of those Lindblad generators for which the QMS admits an essentially commutative dilation. In their work, the term “dilation" is more general than quantum Langevin equations. However, within our framework, we are able to obtain their result, as is stated in the following Theorem.

Theorem III.3. *Let $P_\cdot$ be a QMS on $\mathcal{B}(\mathcal{H})$. Then the following are equivalent.*

1) The semigroup $P_\cdot$ admits a dilation $U_\cdot$, solution of a unitary quantum Langevin equation, such that $\mathcal{A}_t(U_\cdot)$ is commutative, for all t.

2) There exist

– selfadjoint operators $H, L_1, ..., L_m$ on $\mathcal{H}$,

– unitary operator $S_1, ..., S_n$ on $\mathcal{H}$,

– positive real numbers $\lambda_1, ..., \lambda_n$,

such that the Lindblad generator $\mathcal{L}$ of $\mathcal{P}_\cdot$ is given by:

$$\mathcal{L}(X) = -i[H, X] + \frac{1}{2}\sum_{k=1}^{m}\left(2L_k X L_k - L_k^2 X - X L_k^2\right) + \sum_{k=1}^{n}\lambda_i\left(S_k^* X S_k - X\right). \qquad \text{(III.17)}$$

Proof. The proof is an immediate consequence of Theorem III.1. Indeed, if $\mathcal{P}_\cdot$ admits a dilation $U_\cdot$ such that $\mathcal{A}_t(U_\cdot)$ is commutative for all t, then by Theorem III.1 a change of noise leads to a quantum Langevin Equation for $U_\cdot$ of the form of Equation (**??**) and the result follows by Equation (III.4).

Conversely, if the Lindblad generator is of the form 2), the quantum Langevin equation with the corresponding coefficients has its algebra $\mathcal{A}_t(U)$ commutative for all t. □

The other link with QMS we want to mention concerns the detailed balance condition as defined in [FU07]. When the invariant state is the normalized trace, this condition summarizes into

$$\mathrm{Tr}\left[\mathcal{L}(X)Y\right] - \mathrm{Tr}\left[X\mathcal{L}(Y)\right] = \mathrm{Tr}\left[XYK - YXK\right] \text{ for all } X, Y \in \mathcal{B}(\mathcal{H}),$$

where $K \in \mathcal{B}(\mathcal{H})$ is a selfadjoint operator. Fagnola and Umanita proved in [FU07] that detailed balance condition with respect to the normalized trace holds if and only if there exists a representation of the Limblad generator such that $L_k = L_k^*$ for all $k \in \Lambda$. As a direct consequence of Theorem III.1 we obtain:

Theorem III.4. *A QMS satisfies the detailed balance condition with respect to the normalized trace on $\mathcal{B}(\mathcal{H})$ if and only if its admits a unitary dilation which is the solution of a classical Langevin Equation with Brownian noises only.*

IV Classical and Quantum parts of the Environment

We are now back to general quantum Langevin equation and we shall prove that they can be always splitted into a maximal commutative part and a purely quantum one.

IV. 1 The Decomposition Theorem

In this section we study the Noise Algebra $\mathcal{A}^t(U_\cdot)$ in the general case. Note that if $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$, then by the exponential property of the symmetric Fock space one has

$$\Phi = \Gamma_s\left(L^2(\mathbb{R}^+, \mathcal{K}_1)\right) \otimes \Gamma_s\left(L^2(\mathbb{R}^+, \mathcal{K}_2)\right).$$

For short, if $\tilde{\mathcal{K}}$ is a subspace of $\mathcal{K}$, we write $\Phi(\tilde{\mathcal{K}}) = \Gamma_s\left(L^2(\mathbb{R}^+, \tilde{\mathcal{K}})\right)$. Suppose that $U_\cdot$ is the unitary solution of Equation (III.1) and both $\mathcal{H} \otimes \mathcal{K}_1$ and $\mathcal{H} \otimes \mathcal{K}_2$ are stable by $\mathbb{S}$. Then, as already mentioned before, $\Lambda = \Lambda_1 \cup \Lambda_2$ accordingly to the decomposition of $\mathbb{S}$ and Equation (III.1) can be written as:

$$dU_t = -\left(iH + \frac{1}{2}\sum_{i\in\Lambda}(L_i^0)^* L_i^0\right)U_t\,dt + \sum_{i,j\in\Lambda_1\cup\{0\}} L_j^i U_t da_j^i(t) + \sum_{i,j\in\Lambda_2\cup\{0\}} L_j^i U_t da_j^i(t). \qquad \text{(IV.1)}$$

Definition IV.1. *Let $\mathcal{K}_1$ be a subspace of $\mathcal{K}$ and write $\mathcal{K}_2 = \mathcal{K}_1^\perp$. We say that $\Phi(\mathcal{K}_1)$ is* a Commutative Subsystem of the Environment *if $\mathcal{K}_1 \neq \{0\}$ and:*

- *both $\mathcal{H} \otimes \mathcal{K}_1$ and $\mathcal{H} \otimes \mathcal{K}_2$ are stable by $\mathbb{S}$. Consequently, up to a change of noise, Equation (III.1) takes the form of Equation (IV.1).*
- *$\mathcal{A}_t(U^1)$ is commutative, where $U_\cdot^1$ is the unique unitary solution of the HP Equation:*

$$dU_t^1 = -\frac{1}{2}\sum_{i\in\Lambda_1}\left((L_i^0)^* L_i^0\right)U_t^1\,dt + \sum_{i,j\in\Lambda_1\cup\{0\}} L_j^i U_t^1 da_j^i(t).$$

Consequently, using the notation of the previous definition, if $\Phi(\mathcal{K}_1)$ is a commutative Subsystem of the Environment then Theorem III.1 can be applied to $\tilde{U}_\cdot$, so that it obeys a classical stochastic differential equation driven by independent Poisson processes and Brownian Processes. This in turn implies conditions on the coefficients in Equation (III.1).

Theorem IV.1 (Decomposition Theorem)**.** *Suppose that $U_\cdot$ is the unique unitary solution of Equation (II.6). Then $\mathcal{K}$ is the orthogonal direct sum of two subspace $\mathcal{K}_c$ and $\mathcal{K}_q$, such that either $\mathcal{K}_c = \{0\}$, or:*

- *$\Phi(\mathcal{K}_c)$ is a Commutative Subsystem of the Environment.*
- *If $\tilde{\mathcal{K}}$ is a subspace of $\mathcal{K}$ such that $\Phi(\tilde{\mathcal{K}})$ is a Commutative Subsystem of the Environment, then $\tilde{\mathcal{K}}$ is a subspace of $\mathcal{K}_c$.*
- *$U^q_\cdot$ does not have any Commutative Subsystem, where $U^q_\cdot$ is the unique unitary solution of the HP Equation:*

$$dU_t^q = -\frac{1}{2}\sum_{i\in\Lambda_q}\left((L_i^0)^* L_i^0\right) U_t^q\, dt + \sum_{i,j\in\Lambda_q\cup\{0\}} L_j^i U_t^q da_j^i(t).$$

Proof. The first step of the proof is to identify the subspace $\mathcal{K}_c$. To do that, let $\mathcal{P}_c$ be the set of orthogonal projections $P \in \mathcal{A}(\mathbb{S})'$ such that $P \in \mathcal{P}_c$ iff $\Phi(P\mathcal{K})$ is a Commutative Subsystem of the Environment. We claim that $\mathcal{P}_c$ has a maximal element. Indeed, as $\mathcal{K}$ is finite dimensional, for any totally order set $\mathcal{P} \subset \mathcal{P}_c$ there exists a projection $P_{\max} \in \mathcal{P}$ such that $P_{\max}\mathcal{K}$ has the highest dimension for this set. Consequently $P \leq P_{\max}$ for all $P \in \mathcal{P}$. Thus $\mathcal{P}_c$ is an inductive set and by Zorn Lemma it has a maximal element that we write P_c. We take $\mathcal{K}_c = P_c\mathcal{K}$.

Suppose now that $P_c \neq 0$. By definition, $\Phi(\mathcal{K}_c)$ is a Commutative Subsystem of the Environment. Furthermore, if $\tilde{\mathcal{K}}$ is a subspace of $\mathcal{K}$ such that $\Phi(\tilde{\mathcal{K}})$ is a Commutative Subsystem of the Environment, then the orthogonal projection on $\tilde{\mathcal{K}}$ is dominated by P_c so that $\tilde{\mathcal{K}}$ is a subspace of $\mathcal{K}_c$. Consequently $U^q_\cdot$ does not have any Commutative Subsystem. □

Remarks IV.1. We emphasize that this Theorem states the existence of such a decomposition. However it does not provide any practical way to explicit it. Indeed, the first step in order to find the decomposition is the study the Environment Algebra $\mathcal{A}(\mathbb{S})$. For small matrices this can be done for instance numerically. For instance, in [Bar15] proved that $\mathcal{A}(\mathbb{S})'$ is the eigenspace for the eigenvalue 1 of a certain completely positive map on $\mathcal{K}$. This provides by Theorem II.2 a decomposition of $\mathbb{S}$ as $\mathbb{S} = \mathbb{S}_1 + \mathbb{S}_2$, such that $\mathbb{S}_1$ is the maximal block-diagonal unitary operator that we can extract from $\mathbb{S}$. However this does not give the decomposition of $\Phi(\mathcal{K})$ into a classical and a quantum part.
In the next subsection we develop this point with several example.

IV. 2 Examples and open problems

In the first example below we prove that the spontaneous emission whose evolution is given by Equation I.4 has a purely quantum environment.

Examples IV.1 (Spontaneous Emission)**.** Take $\mathcal{H} \approx \mathbb{C}^2$ and $\mathcal{K} \approx \mathbb{C}$. Let $U_\cdot$ be the solution of the quantum Langevin Equation

$$dU_t = -\frac{1}{2}V^*V\,U_t\,dt + V\,U_t\,da_1^0(t) - V^*\,U_t\,da_0^1(t),$$

where

$$V = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Here $\mathbb{S} = I_{\mathcal{H}}$, so that $\mathcal{A}(\mathbb{S})$ is commutative. However, clearly there does not exist $\lambda \in \mathbb{C}$ such that $V^* = \lambda V$, so by Theorem III.1, $U_\cdot$ has a purely quantum environment.

We rely on this typical evolution in order to construct one example where the decomposition is explicit.

Examples IV.2 (An explicit decomposition)**.** Take $\mathcal{H} \approx \mathbb{C}^2$ and $\mathcal{K} \approx \mathbb{C}^2$. We consider the following coefficients in the quantum Langevin Equation, with $\lambda, \theta \in \mathbb{R}$, $\lambda > 0$:

$$\mathbb{S} = \begin{pmatrix} \sin^2\theta & \cos^2\theta & \sin\theta\cos\theta & \sin\theta\cos\theta \\ \cos^2\theta & \sin^2\theta & \sin\theta\cos\theta & \sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & -\cos^2\theta & \sin^2\theta \\ \sin\theta\cos\theta & -\sin\theta\cos\theta & \sin^2\theta & -\cos^2\theta \end{pmatrix},$$

$$L_1^0 = \begin{pmatrix} -\lambda\cos\theta & \lambda\cos\theta + \sin\theta \\ \lambda\cos\theta & -\lambda\cos\theta \end{pmatrix}, \qquad L_2^0 = \begin{pmatrix} -\lambda\sin\theta & \lambda\sin\theta - \cos\theta \\ \lambda\sin\theta & -\lambda\sin\theta \end{pmatrix}.$$

At first sight, $\mathcal{S}$ is not block-diagonal. However, after the change of noise given by the unitary operator

$$W = \begin{pmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{pmatrix},$$

we find in the new basis:

$$\mathbb{S} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad L_1^0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad L_2^0 = \begin{pmatrix} -\lambda & \lambda \\ \lambda & -\lambda \end{pmatrix} = \lambda(S - I_{\mathcal{H}}),$$

where $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Consequently the quantum Langevin Equation takes the form:

$$dU_t = -\frac{1}{2}\left((L_1^0)^* L_1^0 + (L_2^0)^2\right) U_t\, dt + L_1^0 U_t\, da_1^0(t) - (L_1^0)^* U_t\, da_0^1(t) + L_2^0 U_t\, dX_t,$$

where X_t is a compensated Poisson process of intensity λ and jumps $1/\lambda$.

In this example, finding the decomposition resumes to the decomposition of $\mathbb{S}$, because it is possible to check each coefficients L_i^0 separately after diagonalization of $\mathbb{S}$. This is not always possible.

Examples IV.3 (A non-explicit decomposition)**.** Take $\mathcal{H} \approx \mathbb{C}^2$ and $\mathcal{K} \approx \mathbb{C}^2$. We assume in this example that $\mathbb{S} = I_{\mathcal{H}\otimes\mathcal{K}}$. Consequently the matrix of $\mathbb{S}$ does not depend on the choice of the basis of $\mathcal{K}$. However the coefficients L_i^0 do, $i = 1, 2$. If we take for instance:

$$L_1^0 = \begin{pmatrix} -\lambda\cos\theta & \lambda\cos\theta + \sin\theta \\ \lambda\cos\theta & -\lambda\cos\theta \end{pmatrix}, \qquad L_2^0 = \begin{pmatrix} -\lambda\sin\theta & \lambda\sin\theta - \cos\theta \\ \lambda\sin\theta & -\lambda\sin\theta \end{pmatrix},$$

there is no practical way to find the unitary operator W as in the previous example that gives the decomposition.

Part B

Subclassical Dynamics

The next two chapters focus on classical dynamics that can emerge inside the quantum dynamics of the system.

Chapter 3 was chronologically the first work in my thesis. It focuses on extensions of classical Markovian dynamics to quantum dynamics. The main idea was to use an atypical kind of dilation of classical Markov operators, introduced by Stéphane Attal in [Att10], to construct such extensions. The main result is the proof of an extension for Lévy processes.

Chapter 4 concerns a recent collaboration with Denis Bernard and Yan Pautrat, on some geometric aspect of the classical processes induced by Open Quantum Walks. In particular, we define and solve a Dirichlet problem associated to these processes. My contribution in this work consisted in resolving this problem by using a variational approach based on non-commutative Dirichlet forms.

- Chapter 3 has been submitted to *Stochastic Analysis and Applications*.
- Chapter 4 is part of an article in preparation, in collaboration with Denis Bernard and Yan Pautrat.

Chapter 3

Quantum extensions of dynamical systems and of Markov semigroups

Abstract:
We investigate some particular completely positive maps which admit a stable commutative von Neumann subalgebra. The restriction of such maps to the stable algebra is then a Markov operator. In the first part of this article, we propose a recipe in order to find a quantum extension of a given Markov operator in the above sense. We show that the existence of such an extension is linked with the existence of a special form of dilation for the Markov operator studied by Attal in [Att10], reducing the problem to the extension of dynamical system. We then apply our method to the same problem in continuous time, proving the existence of a quantum extension for Lévy processes. In the second part of this article, we focus on the case where the commutative algebra is isomorphic to $\mathcal{A} = l^\infty(1, ..., N)$ with N either finite or infinite. We propose a classification of the CP maps leaving $\mathcal{A}$ stable, producing physical examples of each classes.

I Introduction

We are interested in those quantum dynamics, on the algebra of bounded operators on some separable Hilbert space, that admit a stable commutative subalgebra. The interest of such a property holds in the fact that if this algebra is maximal, then it can be looked at as the algebra of essentially bounded functions on some measurable space. Hence the restriction to this subalgebra is a semigroup of positive and identity preserving operators, that is a classical Markov semigroup. Consequently, the evolution of the observables of this algebra follow a classical evolution. For this reason, we call such dynamics *subclassical*.

The question of the existence of such a stable algebra was first raised and motivated by Rebolledo in [Reb05] and then studied by several authors [FS07] [RFS08]. We shall be concerned with the converse problem: given a classical Markov semigroup, is it the restriction of a quantum Markov semigroup to a maximal stable commutative algebra? The answer to this question would help to understand which classical processes can appear in quantum dynamics.

There has already been a lot of works on the existence of such quantum extensions. The first motivation was to show the possibility to embed the classical theory of stochastic calculus in the quantum one, created by Hudson and Parthasarathy in their pioneering article [HP84]. Such an embedding shows that the latter is a true generalization of the former. This fact was already remark by Meyer in [Mey89] in the case of finite Markov chains in discrete time. Meyer's construction was extended to the case of jump processes by Parthasarathy and Sinha in [PS90] who constructed the structure maps of the flow through certain group actions. This was followed by the proof of the existence of quantum extensions for Azéma martingales [CF95] by Chebotarev and Fagnola, and Bessel processes [FM96] by Fagnola and Monte. The main idea behind the two last results was highlighted by Fagnola in [Fag99], where he gives sufficient conditions on the generator of a Quantum Markov Semigroup (QMS) in order to admit a classical restriction. As an application, he proved the existence of a quantum extension for diffusive Markov semigroups. Basically, it amounts to prove that the generator is itself a quantum extension of the generator of a classical Markov semigroup. The subsequent recipe has the great advantage to avoid technical problems such as the continuity of the QMS. The main difficulty consists in finding the adequate quantum generator.

In this article, we propose a different recipe to find quantum extension of a Markovian dynamics. This recipe can be view as a generalization of one used by Gregoratti in [Gre08]. It is based on the existence of a particular form of classical dilation of a Markov operator. In fact, it was shown in [Att10] by Attal that Markov operators on Lusin space all admit dilations in the form of a dynamical system on a product space. This particular case of classical dilations was further studied by Deschamps [Des13], who showed how their limits from discrete to continuous time lead to a specific class of stochastic differential equations.

Under some basic assumptions, we make explicit quantum unitary extensions of those

dynamical systems. We show that the quantum conditional expectation (the trace) of this unitary evolution over a state of the environment is a quantum extension of the primary Markov operator.
Consequently, the main difficulty of our recipe lies in the existence of a classical dilation in the Attal-Gregoratti sense, that fulfills the condition to be extended. Focusing on the continuous-time case, we find sufficient conditions on a Markov semigroup to have such a dilation. Although they are only sufficient and not necessary conditions, we believe that they can highlight the probabilistic nature of the problem. Finally, puting altogether these conditions leads naturally to the proof of the existence of a quantum extension for Lévy processes.

In practice, the stable algebra that appears in physical examples is the algebra generated by a self-adjoint operator of the Hilbert space, i.e. an observable of the system. One of the most famous example is obtained in the weak coupling or Van-Hoove limit, that was put in a rigorous mathematical language by Davies [Dav74] (see also [AK02]). There has already been a lot of studies on the QMS that arise in this limit ([AFH06] for example), called generic QMS. An other example of physically realizable Completely Positive map (CP map) with a stable commutative algebra is given by the composition with the von Neumann measurement operator of any CP map. Those examples appear in the case where the observable generating the algebra has discrete and non-degenerate spectrum. Considering these particular cases, we provide a classification of subclassical CP maps. We also highlight a link between subclassical CP maps and quantum trajectories.

This article is structures as followed. In Section II we introduce our notations and our definitions. The main results about quantum extensions are in Section III, where we explain our recipe to find quantum extensions of classical dynamics, and apply it to Lévy processes. Finally in Section IV, we focus on the simpler case where the stable commutative algebra is isomorphic to $\mathcal{A} = l^\infty(1, ..., N)$ with N either finite or infinite.

II Notations and definitions

In this section we give the set-up of our study. It is mainly the same set-up as described by Rebolledo in [Reb09], however we restrict ourselves to the case where the commutative algebra $\mathcal{A}$ is a von Neumann algebra (and not only a C^* algebra). In Subsection II. 1, we recall the definitions of Markov quantum dynamics for open system and the kind of commutative subalgebra we will consider in the following. In Subsection II. 2, we give our definitions of quantum dynamics with stable commutative subalgebra.

II. 1 Quantum probability background for open quantum systems

We are interested in quantum dynamics acting on the algebra $\mathcal{B}(\mathcal{H})$ of bounded operator on a *separable* Hilbert space $\mathcal{H}$. This algebra physically corresponds to the set of the observables of a quantum system $\mathcal{H}$. When the system is closed, in the Heisenberg point of view of time evolution, evolution of observables is given by a group of $*$-homomorphism $(U_t^* \cdot U_t)_{t\geq 0}$, where $(U_t)_t \geq 0$ is a one parameter group of unitary operators. In this case, we shall talk about a *quantum unitary evolution.*
However, in this article we focus on open quantum systems. Quantum Markov process (QMS) are defined as weakly* continuous semigroups $(\mathcal{P}_t)_{t\geq 0}$ of completely positive maps (CP map) on $\mathcal{B}(\mathcal{H})$, such that $\mathcal{P}_t(\mathbb{1}) = \mathbb{1}$. Either the time is discrete or continuous, we will talk about quantum dynamics or simply dynamics on $\mathcal{B}(\mathcal{H})$.

In the following, all CP maps on $\mathcal{B}(\mathcal{H})$ are normal (that is continuous for the σ-weak topology on $\mathcal{B}(\mathcal{H})$) and identity preserving.

We shall be concerned with quantum dynamics that have a commutative von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$ stable under there action.

Definition II.1. *Let $\mathcal{A}$ be a commutative subalgebra of $\mathcal{B}(\mathcal{H})$. We say that $\mathcal{A}$ is* maximal *if it admits a cyclic vector i.e. a vector $\Omega \in \mathcal{H}$ such that $\mathcal{A}\Omega$ is dense in $\mathcal{H}$.*

If $(E, \mathcal{E}, \mu)$ is a measured space and $f \in L^\infty(E, \mathcal{E}, \mu)$ (we simply write $L^\infty(\mu)$), the multiplication operator M_f on $L^2(\mu)$ is the bounded operator defined by

$$(M_f\ g)(x) = f(x)g(x),\ g \in L^2(\mu),\ x \in E.$$

Proposition II.1. *Let $\mathcal{A}$ be a commutative subalgebra of $\mathcal{B}(\mathcal{H})$. The following three assertions are equivalent:*

(i) $\mathcal{A}$ is maximal;

(ii) $\mathcal{A}$ is unitarily equivalent to the algebra of multiplication operators on a Hilbert space $L^2(E, \mathcal{E}, \mu)$ for some measured space $(E, \mathcal{E}, \mu)$;

(iii) $\mathcal{A} = \mathcal{A}'$, *where* $\mathcal{A}'$ *stands for the commutant of* $\mathcal{A}$.

Proof. See [RS80] □

This proposition tells us that whenever $\mathcal{A}$ is maximal, the Hilbert space $\mathcal{H}$ and the algeabra $\mathcal{A}$ can be identified respectively with $L^2(E,\mathcal{E},\mu)$ for some measured space $(E,\mathcal{E},\mu)$ and the algebra $L^\infty(E,\mathcal{E},\mu)$ of bounded functions on this measured space. Moreover, in the corresponding physical applications, the algebra which is stable under the evolution of the system is often $\mathcal{A}'$. This algebra is commutative if and only if it is equal to $\mathcal{A}$. Thus the assumption that $\mathcal{A}$ is maximal is also needed in order to obtain a classical interpretation of the physically stable algebra.

In the following we will always assume that $\mathcal{A}$ *is maximal, and consequently has a cyclic vector* Ω, *which we chose to be norm* 1.

II. 2 Preliminary definitions

We begin with the definition of *subclassical dynamics*. The system is modeled by the Hilbert space $\mathcal{H} = L^2(E,\mathcal{E},\mu)$ for some measured space $(E,\mathcal{E},\mu)$. The commutative subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ is the algebra of multiplication operators by bounded functions on E:

$$\mathcal{A} = \{M_f,\ f \in L^\infty(\mu)\}\,. \tag{II.1}$$

Thus $\mathcal{A}$ can be identified with $L^\infty(\mu)$.

Definition II.2 (Subclassical CP map)**.** *A completely positive map* $\mathcal{L}$ *on* $\mathcal{B}(\mathcal{H})$ *is called* $\mathcal{A}$-subclassical *if* $\mathcal{A}$ *is stable under the action of* $\mathcal{L}$, *i.e.*

$$\mathcal{L}(\mathcal{A}) \subset \mathcal{A}.$$

Definition II.3 (Subclassical QMS)**.** *A quantum Markov semigroup* $(\mathcal{P}_t)_{t\geq 0}$ *is called* $\mathcal{A}$-subclassical *if* $\mathcal{A}$ *is a stable subalgebra of* $(\mathcal{P}_t)_{t\geq 0}$, *i.e.*

$$\mathcal{P}_t(\mathcal{A}) \subset \mathcal{A} \text{ for all } t \geq 0.$$

Note that if $\mathcal{L}$ is a $\mathcal{A}$-subclassical CP map, there obviously exists a linear operator L acting on $L^\infty(\mu)$ such that for all $f \in L^\infty(\mu)$:

$$\mathcal{L}(M_f) = M_{Lf}.$$

If furthermore $\mathcal{L}$ is an identity preserving and normal CP map, it is easy to verify that L is a Markov operator on $L^\infty(\mu)$, that is, it satisfies:

i) L is an operator on $L^\infty(\mu)$;

ii) $L\mathbb{1} = \mathbb{1}$;

iii) $Lf \geq 0$ whenever $f \geq 0$;

iv) $f \mapsto Lf$ is σ-weakly continuous.

In the same way, if $(\mathcal{P}_t)_{t\geq 0}$ is a $\mathcal{A}$-subclassical normal QMS, there exists a Markov semigroup $(P_t)_{t\geq 0}$ on $L^\infty(\mu)$, i.e. P_t are Markov operators for all $t \geq 0$, $P_0 f = f$ on $L^\infty(\mu)$, $t \mapsto P_t$ is strongly continuous on $L^\infty(\mu)$ and for all $f \in L^\infty(\mu)$:

$$\mathcal{P}_t(M_f) = M_{P_t f} \text{ for all } t \geq 0.$$

In the next section, we will focus on the following problem: given a Markov dynamics, either in discrete or continuous time, is it the restriction of a quantum dynamics on a stable commutative algebra. We will use the following definition, which naturally extend to the continuous case.

Definition II.4. *Let $(E, \mathcal{E}, \mu)$ be a probability space, and let L be a Markov operator on $L^\infty(\mu)$. A CP map $\mathcal{L}$ on $\mathcal{B}(L^2(\mu))$ is called a* ***quantum extension*** *of L if for all $f \in L^\infty(\mu)$,*

$$\mathcal{L}(M_f) = M_{Lf}, \tag{II.2}$$

i.e. $\mathcal{L}$ is $\mathcal{A}$-subclassical with $\mathcal{A} = \{M_f,\ f \in L^\infty(\mu)\}$ and the classical restriction of $\mathcal{L}$ to $\mathcal{A}$ is L.

III Subclassical dynamics: the dynamical system point of view

In this section we show how quantum extensions of Markov operators can be obtained by a general scheme. This scheme relies on the following remarks:

1) it is easy to give conditions in order to find quantum extensions of dynamical systems;

2) dynamical systems naturally appear as dilation of Markov operators in a certain form;

3) finally one can recover a quantum extension of a Markov operator via the quantum extension of its dilation.

$$
\begin{array}{ccc}
\text{Quant. ext. of the Markov operator} & \xleftarrow{\ 3)\ } & \text{Quant. ext. of the dynamical system} \\
\uparrow & & \uparrow \scriptstyle{1)} \\
\text{Markov operator} & \xrightarrow{\ 2)\ } & \text{Dynamical system}
\end{array}
$$

In Subsection III. 1, we focus on point 1). We find conditions so that a dynamical system admits a quantum *unitary* extension and under these conditions we completely characterize such possible extensions. In Subsection III. 2, we explain point 2) and 3) and we prove that the quantum extension does not depend on the choice of the classical dilation in Subsection III. 3. In Subsection III. 4 we apply our recipe for the case of finite states spaces.

Finally in Subsection III. 5 we treat the case of continuous time Markov semigroups, which allows us to prove the existence of a quantum extension for Lévy processes in Subsection III. 6.

III. 1 Subclassical unitary evolutions as quantum extensions of dynamical systems

In this section we look for conditions on a dynamical system that allow the existence of its quantum extension. We first emphasize in Proposition III.1 that unitary evolutions are natural candidates for quantum extension of dynamical system. This leads us to find appropiate conditions on a dynamical system to admits a quantum extension given by a unitary evolution (that is a quantum unitary extension). We do that in Proposition III.2. Finally in Proposition III.3 we characterize all quantum unitary extensions.

A dynamical system is a quadruplet $(E, \mathcal{E}, \mu, T)$, where $(E, \mathcal{E}, \mu)$ is some measured space, and $T : E \to E$ is a $\mathcal{E}$-measurable function. It can be seen as a Markov operator. Indeed, we will say that a Markov operator L on the measured space $(E, \mathcal{E}, \mu)$ is deterministic if there exist a measurable function T on $(E, \mathcal{E}, \mu)$ such that for all $f \in L^\infty(\mu)$, we have:

$$Lf = f \circ T.$$

Proposition III.1. *Suppose $(E, \mathcal{E}, \mu)$ is a Lusin space. Then the restriction of an $\mathcal{A}$-subclassical unitary evolution on $L^2(\mu)$ to its classical part is deterministic.*

Proof. Let U be a unitary operator on $L^2(\mu)$. We suppose that the corresponding unitary evolution is $\mathcal{A}$-subclassical, i.e. there exists a Markov operator L on $L^\infty(\mu)$ such that for all $f \in L^\infty(\mu)$,

$$U^* M_f U = M_{Lf}.$$

For all $f, g \in L^\infty(\mu)$, we have $M_{Lf} M_{Lg} = M_{Lfg}$ and $M_{L\overline{f}} = M_{\overline{Lf}}$. Consequently, L is a $*$-homomorphism of the algebra $L^\infty(\mu)$. As was shown by Attal in [Att10], under the hypothesis that $(E, \mathcal{E}, \mu)$ is a Lusin space, Markov operators that are $*$-homomorphism of the algebra $L^\infty(\mu)$ are exactly the deterministic ones. □

This proposition tells us that if U is a unitary operator on $\mathcal{H}$ and if the corresponding evolution on $\mathcal{B}(\mathcal{H})$ is $\mathcal{A}$-subclassical, then there exists a measurable application $T : E \to E$ such that for every $f \in L^\infty(\mu)$,

$$U^* M_f U = M_{f \circ T}, \tag{III.1}$$

i.e. $U^* \cdot U$ is a quantum extension of the dynamical system $(E, \mathcal{E}, \mu, T)$. Consequently from Proposition III.1 unitary evolutions appears as natural candidates for quantum extensions of dynamical systems. This remark is made effective in the next proposition.

Proposition III.2. *Let $(E, \mathcal{E}, \mu, T)$ be a dynamical system. Assume that T is invertible and μ-preserving. Then the operator $U_T : L^2(\mu) \to L^2(\mu)$ defined by*

$$U_T f = f \circ T, \tag{III.2}$$

is a unitary operator, and we have $U_T^* = U_{T^{-1}}$*. Moreover, for all* $f \in L^\infty(\mu)$*, we have*

$$U_T M_f U_{T^{-1}} = M_{f\circ T}. \tag{III.3}$$

Proof. The first part of the proposition is the well-known Koopman's Lemma (see [RS80]). We prove the second part. For all $f \in L^\infty(\mu)$ and $g \in L^2(\mu)$ we have:

$$\begin{aligned} U_T M_f U_{T^{-1}} g &= U_T M_f (g \circ T^{-1}) \\ &= U_T \left[f\left(g \circ T^{-1}\right)\right] \\ &= \left[f(g\circ T^{-1})\right] \circ T \\ &= (f \circ T) g \\ &= M_{f\circ T} g. \end{aligned}$$

□

Thus under the condition that T is invertible and μ-preserving, the unitary evolution given by $U_{T^{-1}}$ is a quantum extension of T. From now on, the choice of this quantum extension will be called the *canonical quantum extension*. We finish this section by a characterization of all quantum unitary extensions of invertible and measure-preserving dynamical systems.

Proposition III.3. *Let* T *be an invertible and* μ*-preserving dynamical system on* $(E, \mathcal{E}, \mu)$*. Let* $U = U_{T^{-1}}$ *be the canonical quantum extension of* T*. Let* V *be a unitary operator on* $L^2(\mu)$*. Then the two following assertions are equivalent:*

(i) for all $f \in L^\infty(\mu)$*,* $V^* M_f V = M_{f\circ T}$*;*

(ii) there exists $g \in L^\infty(\mu)$ *such that* $V = M_g\, U$*.*

Furthermore, if this is realized, then g *in (ii) is such that* $|g|^2 = 1$ μ*-almost surely.*

Proof. The last point of the theorem is a straightforward computation.
Recall that $\mathcal{A}$ is the algebra of multiplication operators by bounded functions. Then (ii) is equivalent to $VU^* \in \mathcal{A} = \mathcal{A}'$, which itself is equivalent to $VU^* M_f = M_f V U^*$ for all $f \in L^\infty(\mu)$. Multiplying to the right by V^* and to the left by U we get that (ii) is equivalent to $V^* M_f V = U^* M_f U$ for all $f \in L^\infty(\mu)$, which is (i). □

Consequently quantum extensions of an invertible and measure-preserving dynamical system $(E, \mathcal{E}, \mu T)$ are in one-to-one correspondence with elements $g \in L^\infty(\mu)$ of constant modulus 1.

III. 2 Subclassical CP maps as quantum extensions of classical dilations

As was shown by Gregoratti in [Gre09] for the case of finite spaces and latter by Attal in [Att10] in the general case, classical dynamical systems appear as natural dilations of Markov operators: the latter can be written as the "trace" of a deterministic evolution on a larger space.
As a second step of our recipe, in this section we show that the trace over a state of the environment of a unitary quantum extension leads to a quantum extension of the Markov operator.
First we recall a result obtain by Attal in [Att10].

Let $(E, \mathcal{E}, \mu)$ be a measured space, μ being not necessarily finite. This space stands for the small system. Let $(F, \mathcal{F}, \nu)$ be a probability space, which stands for the environment. Let T be an $\mathcal{E} \otimes \mathcal{F}$-measurable application from $E \times F$ to $E \times F$. The space $(E \times F, \mathcal{E} \otimes \mathcal{F}, \mu \otimes \nu, T)$ is a dynamical system, which describes the evolution of the whole system.

In the Heisenberg description of time evolution of a physical system, what we consider is the evolution of observables rather than the evolution of the state of the system. In the commutative case observables are functions $h \in L^\infty(\mu \times \nu)$, and their evolution is given by the operator $\tilde{T}$ on $L^\infty(\mu \otimes \nu)$ defined by $\tilde{T} : h \mapsto h \circ T$.

Suppose now that we only have access to the space E and the action of the environment is only given by the expectation over the measure ν on $(F, \mathcal{F})$. Starting at a deterministic point $x \in E$ and at a random point $y \in F$ with law ν, one step evolution of an observable $f \in L^\infty(\mu)$ is given by the operator L on $L^\infty(\mu)$ given by:

$$Lf(x) = \int_F \tilde{T}(f \otimes \mathbb{1})(x, y) d\nu(y), \ f \in L^\infty(\mu), \ x \in E. \tag{III.4}$$

Proposition III.4 (Theorem 2.2 in [Att10]). *The operator L is a Markov operator on $L^\infty(E)$.*
Conversely, for any Markov operator on a Lusin *space $(E, \mathcal{E}, \mu)$, one can find a measurable space $(F, \mathcal{F})$, a probability measure ν on F and an invertible dynamical system $T : E \times F \to E \times F$ such that L is given by Equation (III.4).*

Thus a Markov dynamics can always be seen as the restriction to a subsystem of a deterministic dynamics.

We now come back to the quantum world. The small system is the Hilbert space $\mathcal{H} = L^2(E, \mathcal{E}, \mu)$ and the environment is the Hilbert space $\mathcal{K} = L^2(F, \mathcal{F}, \nu)$. The only information we have on the environment is given by a state ω on $\mathcal{K}$. This state defines on $(F, \mathcal{F})$ a probability measure ν_ω via :

$$\nu_\omega(J) = \mathrm{Tr}[\omega M_{\mathbb{1}_J}], \tag{III.5}$$

where $M_{\mathbb{1}_J}$ is the multiplication operator by the characteristic function $\mathbb{1}_J$ of the set $J \in \mathcal{F}$.

Theorem III.1. *Let $T : E \times F \to E \times F$ be a dynamical system. Suppose that there exists a quantum extension of T on $\mathcal{H} \otimes \mathcal{K}$ given by a unitary operator U on $\mathcal{H} \otimes \mathcal{K}$, that is, for all $h \in L^\infty(E \times F)$ we have*

$$U^* M_h U = M_{h \circ T}.$$

Let ω be a state on $\mathcal{K}$. Then the application $\mathcal{L}_\omega : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ defined by

$$\mathcal{L}_\omega(X) = Tr_\omega[U^*(X \otimes I)U], \tag{III.6}$$

is a $\mathcal{A}$-subclassical CP map on $\mathcal{B}(\mathcal{H})$, such that the associated Markov operator L_ω is given by

$$L_\omega f(x) = \int_F (f \otimes \mathbb{1}) \circ T(x, y) d\nu_\omega(y), \;\; f \in L^\infty(E). \tag{III.7}$$

Proof. The proof is almost straightforward. The fact that $\mathcal{L}_\omega$ is a normal identity preserving CP map is well-known. Consequently we just have to check that, for all $f \in L^\infty(E)$, we have

$$\mathrm{Tr}_\omega[U^*(M_f \otimes I)U] = M_{L_\omega f},$$

with L_ω given by Equation (III.4). Then, for all $\rho \in \mathcal{L}_1(\mathcal{H})$,

$$\begin{aligned}
\mathrm{Tr}\left\{\mathrm{Tr}_\omega[U^*(M_f \otimes I)U]\rho\right\} &= \mathrm{Tr}\left\{M_{T(f\otimes\mathbb{1})}(\rho \otimes \omega)\right\} \\
&= \int_{E\times F} T(f \otimes \mathbb{1}) d\mu_\rho \otimes d\nu_\omega \\
&= \int_E \left[\int_F T(f \otimes \mathbb{1}) d\nu_\omega\right] d\mu_\rho \\
&= \int_E [L_\omega f] d\mu_\rho \\
&= \mathrm{Tr}[M_{L_\omega f}\rho].
\end{aligned}$$

This shows that $\mathrm{Tr}_\omega[U^*(M_f \otimes I)U] = M_{Lf}$. □

Remarks III.1. For all states ω on $\mathcal{K}$, the probability measure ν_ω is absolutely continuous with respect to ν. Indeed, if ω is a pure state, i.e. $\omega = |\psi\rangle\langle\psi|$ for some $\psi \in \mathcal{K}$, then for all $J \in \mathcal{F}$, $\nu_\omega(J) = \langle\psi, M_{\mathbb{1}_J}\psi\rangle = \int_J |\psi|^2 d\nu$, and obviously $\nu_\omega(J) \ll \nu$. The same holds if ω is a mixed state by diagonalizing it as a mixture of pure states.
Conversely, if $\tilde{\nu}$ is a probability measure on $(F, \mathcal{F})$ which is absolutely continuous with respect to ν, then there exists $\psi \in \mathcal{K}$ such that for all $J \in \mathcal{F}$, $\tilde{\nu} = \int_J |\psi|^2 d\nu$. Consequently taking $\omega = |\psi\rangle\langle\psi|$ leads to $\nu_\omega = \tilde{\nu}$.

In Theorem III.1, the quantum extension of the dynamical system is not necessarily the canonical choice. Furthermore it is not necessary for the dynamical system to be invertible and measure preserving, as soon as it possesses a quantum unitary extension. However, the quantum extension of the Markov operator depends on this choice, as it will not act in same way outside $\mathcal{A}$. The quantum extension associated to the canonical choice for the unitary operator, when it exists, will be called the *canonical choice associated to* T. In the next Subsection, we prove that the canonical choice does not depend on the choice of the dilation T.

III. 3 Uniqueness of the quantum extensions

So far, we have given a general method to construct a quantum extension of a given Markov operator L. We have already noticed that this quantum extension depend on the choice of the quantum unitary extension of the dilation. However we are going to prove that it does not depend on the choice of the classical dilation of L.

Theorem III.2. *Suppose* $(E\times F_1, \mathcal{E}\otimes\mathcal{F}_1, \mu\otimes\nu_1, T_1)$ *and* $(E\times F_2, \mathcal{E}\otimes\mathcal{F}_2, \mu\otimes\nu_2, T_2)$ *are both invertible and measure-preserving dilations of* L*. Then the* canonical *quantum extension associated to each dilation are the same.*

Proof. We define the CP map $|L\rangle$ associated to L:

$$\begin{array}{cccc} |L\rangle : & L^\infty(\mu) & \to & \mathcal{B}(L^2(\mu)). \\ & f & \mapsto & M_{Lf} \end{array}$$

Let $(E \times F_i, \mathcal{E}\otimes\mathcal{F}_i, \mu\otimes\nu_i, T_i)$, $i = 1, 2$, be two classical dilations of L such that the T_i's are invertible and $\mu\otimes\nu_i$-preserving. Two Stinespring representations [Fag99] of $|L\rangle$ are given by $(L^2(\mu\otimes\nu_i), V_i)$, where the operators V_i are defined by

$$\begin{array}{cccc} V_i : & L^2(\mu) & \to & L^2(\mu\otimes\nu_i) \\ & h & \mapsto & (h\otimes\mathbb{1})\circ T_i^{-1} \end{array} .$$

Then we have for all $f \in L^\infty(\mu)$:

$$|L\rangle f = V_i^* M_f \otimes I_{\mathcal{K}_i} V_i.$$

As the T_i's are invertible and measure-preserving, the V_is are isometric. Furthermore, the canonical quantum extension associated to each dilation is

$$\mathcal{L}_i(\cdot) = V_i^*(\cdot \otimes I)V_i.$$

We can assume that both Stinespring representations are *minimal*, which means that the vector fields spanned by the sets $\{M_f \otimes I\ V_i h,\ f \in L^\infty(\mu),\ h \in L^2(\mu)\}$ are total in $L^2(\mu\otimes\nu_1)$ and $L^2(\mu\otimes\nu_2)$ respectively. Consequently (see [Fag99] for example) there exists a unitary operator $W : L^2(\nu_1) \to L^2(\nu_2)$ such that for all $f \in L^\infty(\mu)$ and $h \in L^2(\mu)$

$$I \otimes W\,(M_f \otimes I\ V_1\ h) = M_f \otimes I\ V_2\ h.$$

In particular, $I \otimes W V_1 = V_2$ and for all $X \in \mathcal{B}(L^2(\mu))$,

$$\begin{aligned}
\mathcal{L}_1(X) &= V_1^*(X \otimes I)V_1 \\
&= V_2^*\,(I \otimes W\ \ X \otimes I\ \ I \otimes W)\,V_2 \\
&= V_2^*(X \otimes I)V_2 \\
&= \mathcal{L}_2(X).
\end{aligned}$$

□

We are now going to apply Theorem III.1 to the case of discrete states spaces.

III. 4 Quantum unitary extensions of dynamical systems for discrete states spaces

The existence of a quantum extension for discrete states spaces was already proved by Parthasarathy and Sinha in [PS90] using quantum stochastic calculus. It was further study by Gregoratti in [Gre08] and [Gre09] as an application of Theorem (III.1). In this section we want to emphasize that reversibility of the dilation is not only a sufficient but also a necessary condition. For discrete spaces, the measure-preserving condition is no longer needed as it is implied by the reversibility condition.

Theorem III.3. *Suppose $E = \{1, ..., N\}$ is a finite or countable state space, endowed with its full σ-algebra. Let $T : E \to E$ be a dynamical system on it. Then T admits a*

quantum unitary extension on $l^2(E)$ if and only if it is invertible.
In this case, the unitary operator U defined by

$$\langle e_x, U e_y \rangle = \delta^x_{T(y)}, \tag{III.8}$$

gives the canonical quantum extension.

Proof. Suppose there exists a quantum unitary extension of T given by a unitary operator U on $\mathcal{H}$. We decompose U in the canonical basis $(e_x)_{x\in E}$ of $\mathcal{H} = l^2(E)$:

$$U = \sum_{x,y\in E} u_{x,y} |x\rangle\langle y|.$$

Then for all $x \in E$, we have

$$\begin{aligned}
\sum_{y\in E} \delta^x_{T(y)} |e_y\rangle\langle e_y| &= M_{\mathbb{1}_{T^{-1}(\{x\})}} \\
&= U^* M_{\mathbb{1}_{\{x\}}} U \\
&= U^* |e_x\rangle\langle e_x| U \\
&= |U^* e_x\rangle\langle U^* e_x| \\
&= \sum_{y\in F} |u_{x,y}|^2 |e_y\rangle\langle e_y|
\end{aligned}$$

Thus for all $x, y \in E$, we have $|u_{x,y}|^2 = \delta^x_{T(y)}$. As U is unitary, for all $x \in E$,

$$\sum_{y\in E} |u_{x,y}|^2 = 1.$$

This implies that for all $x \in E$, there exists a unique $y \in E$ such that $x = T(y)$, i.e. T is invertible.

Suppose now that such a quantum unitary extension exists and consequently that T is invertible. Thus we are looking for a unitary operator U on $\mathcal{H}$ such that for all $x \in E$:

$$U^* |e_x\rangle\langle e_x| U = |T^{-1}(x)\rangle\langle T^{-1}(x)|. \tag{III.9}$$

Looking at the computation above, a sufficient condition for Equation (III.9) to hold is given by Equation (III.8), i.e. the matrix of U in the orthonormal basis $(e_x)_{x\in E}$ is a permutation matrix (either finite on infinite depending on wether E is finite or infinite). □

Corollary III.1. *Let Q be a stochastic matrix on a discrete space, either finite or countable. Then Q admits a quantum extension.*

Proof. Because of Theorem III.3, it is enough to prove that Q admits a classical dilation in the sense of Equation (III.4), with F discrete and T invertible. However this is always the case (see Theorem 2.3 in [Att10]). □

We shall come back to the case of quantum extension for discrete dynamical systems in Section IV. 3.

III. 5 Quantum extension of Markov semigroups in continuous time

In this section we apply the general scheme developed so far to the case of continuous time.

Let $(P_t)_{t\in\mathbb{R}^+}$ be a Markov semigroup on the space $L^\infty(\mu)$ associated to some measured space $(E,\mathcal{E},\mu)$. We suppose that $(E,\mathcal{E},\mu)=(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d),\lambda)$, where $\mathcal{B}(\mathbb{R}^2)$ is the Borel σ-algebra of $\mathbb{R}^d$ and λ is the Lebesgue measure. We look for a Quantum Markov Semigroup $(\mathcal{P}_t)_{t\geq 0}$ on the Hilbert space $\mathcal{H}=L^2(\mu)$ such that for all $t\geq 0$:

$$\mathcal{P}_t(M_f)=M_{P_tf}. \tag{III.10}$$

Our strategy is the following:

1. find a probability space $(F,\mathcal{F},\nu)$ and a family of invertible and measure-preserving maps $(T_t)_{t\in\mathbb{R}^+}$ such that the dynamical system in continuous time $(E\times F,\mathcal{E}\otimes\mathcal{F},\mu\otimes\nu,(T_t))$ is a dilation of the Markov semigroup $(P_t)_{t\in\mathbb{R}^+}$;

2. extend this dynamical system into a unitary evolution $U_t^*\cdot U_t$ on the Hilbert space $L^2(\mu\otimes\nu)$;

3. take the trace over the quantum environment in order to obtain a family of CP maps $(\mathcal{P}_t)$:
$$\mathcal{P}_t(X)=\mathrm{Tr}_\omega[U_t^*(X\otimes I)U],\ \text{for all } t\geq 0 \text{ and } X\in\mathcal{B}(\mathcal{H}); \tag{III.11}$$

4. ensure that this family is a QMS.

We resume this strategy on the following diagram:

$$\begin{array}{ccc}
(\mathcal{P}_t) & \xleftarrow{\ 3)\ } & (U_t^*\cdot U_t) \\
\Big\downarrow{\scriptstyle 4)} & & \Big\downarrow{\scriptstyle 2)} \\
(P_t) & \xrightarrow{\ 1)\ } & (T_t)
\end{array}$$

Notice that we would obtain the same commutative diagram as in the previous section. Though, we get the following result. Notice that the main difference is that a family $(\mathcal{P}_t)_{t\geq 0}$ such that Equation (III.10) holds is not necessarily a QMS.

Theorem III.4. *Under the following conditions:*

A1) (P_t) *is* μ*-preserving;*

A2) (P_t) *is the semigroup of transition probabilities associated to a Markov process* $(X_t)_{t\geq 0}$ *on* $(E, \mathcal{E})$ *with stationary and independent increments;*

then there exists a semigroup of unital CP maps $(\mathcal{P}_t)_{t\geq 0}$ *such that Equation (III.10) holds. If furthermore:*

B) [Continuity condition] For all $\psi \in L^2(\mu)$,

$$\mathbb{E}[|\psi(X_t) - \psi(X_0)|^2] \xrightarrow{t\to 0} 0; \tag{III.12}$$

then the semigroup $(\mathcal{P}_t)_{t\geq 0}$ *is weakly*-continuous and consequently it is a quantum Markov semigroup.*

Proof. We divide the proof of the theorem into three parts. First, under condition A.1) and A.2), we prove the existence of a family of CP map $(\mathcal{P}_t)$ which satisfy Equation (III.10). This is done using the same method as in discrete time. However, the family we shall obtain is not a one parameter semigroup. In the second part of the proof we show how to modify this family in order to obtain a semigroup. Finally, in the last part of the proof, we prove the weak*-continuity of the quantum extension under condition B).

Existence of a family of CP maps such that Equation (III.10) holds: Under Condition A2), there exists a Markov process $(X_t)_{t\geq 0}$ on $(E, \mathcal{E})$ such that for all $f \in L^\infty(E)$ and $s, t \in \mathbb{R}^+$

$$P_t f(X_s) = \mathbb{E}[f(X_{s+t})|X_s]. \tag{III.13}$$

The Markov process is associated to the canonical probability space $(G, \mathcal{G}, \mathbb{P}_\mu)$ where:

- $G = E^{\mathbb{R}^+}$;
- $\mathcal{G}$ is the σ-algebra on G induced by cylindrical measurable sets of E;

- $\mathbb{P}_\mu$ is the law of the Markov process. It can be viewed as the probability measure given by the Kolmogorov Consistency Theorem when considering the laws of the marginals of (X_t).

Thus μ is understood as the law of the random variable X_0 while $\mathbb{P}_\mu$ is the law of the whole Markov process (X_t), defined through its marginals by the Kolmogorov Consistency Theorem.

As (X_t) is stationary with independent increments, the measure $\mathbb{P}_{\delta_0}$, where δ_0 is the Dirac measure at 0, is also the law of the process $(X_t - X_0)$, so we have $\mathbb{P}_\mu = \mu \otimes \mathbb{P}_{\delta_0}$. Consequently, the space $(G, \mathcal{G}, \mathbb{P}_\mu)$ can be factorized as $(E \times F, \mathcal{E}, \otimes \mathcal{F}, \mu \otimes \mathbb{P}_{\delta_0})$, where $(F, \mathcal{F}, \mathbb{P}_{\delta_0})$ is the canonical space for the process $(X_t - X_0)$.
We now define on $(E \times F, \mathcal{E}, \otimes \mathcal{F}, \mu \otimes \mathbb{P}_{\delta_0})$ the family of left shifts $(\theta_t)_{t \geq 0}$ by

$$\begin{array}{llll} \theta_t : & E \times F & \to & E \times F \\ & (x, (\omega_s)_{s \in \mathbb{R}}) & \mapsto & (x + \omega_t, (\omega_{s+t} - \omega_s)_{s \geq 0}) \end{array} \tag{III.14}$$

It is easy to check that it is a semigroup. A simple computation shows that the dynamical system $(E \times F, \mathcal{E}, \otimes \mathcal{F}, \mu \otimes \mathbb{P}_{\delta_0}, (\theta_t))$ is a dilation of the semigroup (P_t), for Equation (III.13) can be written as

$$P_t f = \int_F (f \otimes \mathbb{1}) \circ \theta_t \, d\mathbb{P}_{\delta_0}. \tag{III.15}$$

Under condition A1), by consistency of the measure $\mathbb{P}_\mu$ with respect to the measure μ, the dynamical system (θ_t) is measure-preserving. However it is not invertible. In order to make the θ_t invertible, we have to enlarge the state space by considering negative time. We do this by considering an independent copy of (X_t), but indexed by negative time. Then we define a new process $((\hat{X})_t)_{t \in \mathbb{R}}$, indexed by the whole real line, which is conditioned to $\hat{X}_0 = 0$ and has a canonical space $(\hat{F}, \hat{\mathcal{F}}, \hat{\mathbb{P}}_{\delta_0})$. Now it is easy to prove that the family of shift $(\hat{\theta}_t)_{t \in \mathbb{R}}$ defined by Equation (III.14), but this time on $(E \times F, \mathcal{E} \otimes \hat{\mathcal{F}}, \mu \otimes \hat{\mathbb{P}}_{\delta_0})$, is a group of invertible and $\mathbb{P}_\mu$-preserving applications.

We adopt the notation notation $(F, \mathcal{F}, \nu) = (\hat{F}, \hat{\mathcal{F}}, \hat{\mathbb{P}}_{\delta_0})$ and we define the Hilbert space $\mathcal{K} = L^2(F, \mathcal{F}, \nu)$. Consequently, the family of operators $(U_t)_{t \in \mathbb{R}}$ defined on $\mathcal{H} \otimes \mathcal{K}$ by:

$$U_t h = h \circ \hat{\theta}_t^{-1} \text{ for all } h \in \mathcal{H} \otimes \mathcal{K}, \tag{III.16}$$

is a one-parameter group of unitary operators. Because of Theorem III.1 and Remark III.1, we have the relation

$$\mathrm{Tr}_{|\mathbb{1}\rangle\langle\mathbb{1}|}\left[U_t^*(M_f\otimes I)U_t\right] = M_{P_t f} \text{ for all } t\geq 0.$$

This end the first part of the proof.

Semigroup property of the quantum extension: The left-hand side of the previous equation does not define a semigroup of CP maps on $\mathcal{B}(\mathcal{H})$. However, the family (U_t) is linked in a canonical way to a family of unitary operators (V_t), whose partial trace over the state $|\mathbb{1}\rangle\langle\mathbb{1}|$ of $\mathcal{K}$ gives rise to a one parameter semigroup.

Define on F the family of left shift $(\eta_t)_{t\in\mathbb{R}}$ by $\eta_t : (w_s)\in F\mapsto(w_{t+s}-w_s)$. The family (η_t) is a one-parameter group of ν-preserving and invertible applications, so that the family of operators $(\Theta_t)_{t\in\mathbb{R}}$ on $\mathcal{K}$, defined by $\Theta_t : \psi\in\mathcal{K}\mapsto\psi\circ\eta_t$, is a one parameter unitary group. The operator $I_{\mathcal{H}}\otimes\Theta_t$ acting on $\mathcal{H}\otimes\mathcal{K}$ shall also by written Θ_t. Finally for all $t\geq 0$, define (V_t) and $(\mathcal{P}_t)$ by

$$V_t = \Theta_t^* U_t, \tag{III.17}$$

$$\mathcal{P}_t(X) = \mathrm{Tr}_{|\mathbb{1}\rangle\langle\mathbb{1}|}\left[V_t^* X\otimes I V_t\right]. \tag{III.18}$$

A simple computation shows that $(V_t)_{t\geq 0}$ is a *left-cocycle with respect to the left quantum shift* (Θ_t), i.e. for all $s,t\geq 0$, we have

$$V_{s+t} = \Theta_s^* V_t \Theta_s V_s.$$

It is a well-known result since the paper of Accardi [Acc78] that it implies that $(\mathcal{P}_t)$ has the semigroup property.

Continuity of the quantum extension: We now show that condition B) is a sufficient condition on (P_t) so that $(\mathcal{P}_t)$ is weakly* continuous. Weak* continuity of $(\mathcal{P}_t)$ means that for all $X\in\mathcal{B}(\mathcal{H})$ and all $\rho\in\mathcal{L}_1(\mathcal{H})$,

$$\lim_{t\to 0}\mathrm{Tr}[(\mathcal{P}_t(X)-X)\rho] = 0.$$

As the space of finite rank operators is dense in the predual $\mathcal{L}_1(\mathcal{H})$ of $\mathcal{B}(\mathcal{H})$ for the trace norm, it is enough to prove the last limit for rank-one projectors. Finally it is enough to prove that for all $\psi \in \mathcal{H}$ and $X \in \mathcal{B}(\mathcal{H})$,

$$\lim_{t\to 0} \langle \psi, (\mathcal{P}_t(X) - X)\,\psi\rangle = 0.$$

Remark that the limit (III.12) can be rewritten as $V_t\psi \otimes \mathbb{1} \to \psi \otimes \mathbb{1}$ when $t \to 0$ in $\mathcal{H} \otimes \mathcal{K}$. Fixe $\psi \in \mathcal{H}$ and $X \in \mathcal{B}(\mathcal{H})$. Write $\varphi = \psi \otimes \mathbb{1}$ and $\varphi_t = V_t\psi \otimes \mathbb{1}$. As (φ_t) converge when t goes to 0, it is uniformly bounded in norm for small t, say be $C > \|\psi\|$. Then we have, for t small enough,

$$\begin{aligned}
\big|\langle\psi, (\mathcal{P}_t(X) - X)\psi\rangle\big| &= \big|\mathrm{Tr}\big\{|\psi\rangle\langle\psi|\,(\mathcal{P}_t(X) - X)\,\big\}\big| \\
&= \big|\mathrm{Tr}\big\{|\psi\rangle\langle\psi|\,\big(\mathrm{Tr}_{|\mathbb{1}\rangle\langle\mathbb{1}|}[V_t^* X \otimes I V_t] - X\big)\,\big\}\big| \\
&= \big|\mathrm{Tr}\big\{|\psi\rangle\langle\psi| \otimes |\mathbb{1}\rangle\langle\mathbb{1}|\,[V_t^* X \otimes I V_t - X \otimes I] \\
big\}\big| \\
&= \big|\mathrm{Tr}\big\{\,[|\varphi_t\rangle\langle\varphi_t| - |\varphi\rangle\langle\varphi|]\,X \otimes I\big\}\big| \\
&\le \big|\langle X(\varphi_t - \varphi), \varphi\rangle\big| \\
&\quad + \big|\langle X\varphi_t, (\varphi_t - \varphi)\rangle\big| \\
&\le 2\,\|\varphi_t - \varphi\|\,\|X\|\,C.
\end{aligned}$$

This proves the weak* continuity of the quantum extension, under condition B).

□

III. 6 Quantum extension for Lévy processes

We now apply the result of the previous section to show the existence of a quantum extension for Lévy processes.

Definition III.1. *A Markov process $(Y_t)_{t\ge 0}$ with values in $\mathbb{R}^d$ such that the three following properties hold is called a **Lévy process**.*

(i) $Y_0 = 0$ almost surely;

(ii) (Y_t) is a stationary process with independent increments, i.e. for all $0 \le s \le t$, the random variable $Y_t - Y_s$ is independent from $(Y_u)_{0\le u\le s}$ and has the same law has Y_{t-s};

(iii) Y_t converge in probability to 0 when $t \to 0$.

Theorem III.5. *Let (X_t) be a μ-preserving process, such that $(X_t - X_0)$ is a Lévy process with values in $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \lambda)$. Let (P_t) be its associated semigroup. Then the semigroup of CP maps $(\mathcal{P}_t)$ defined by Equation (III.18) is a quantum extension of (P_t).*

Proof. Let $C_0(\mathbb{R}^d)$ be the set of complex-values continuous function on $\mathbb{R}^d$, which tend to 0 at infinity. Clearly, as a Lévy process is also a Feller process, it is enough to prove the limit (III.12) in the case where $\psi \in C_0(\mathbb{R}^d)$.

Let $\varepsilon > 0$ and $\psi \in C_0(\mathbb{R}^d)$ be fixed. As ψ tends to 0 at infinity, there exists $c > 0$ such that $|\psi(x)| < \varepsilon$ whenever $|x| > c/2$.

As ψ is continuous, it is uniformly continuous on the closed ball of center 0 and radius $2c$, so that there exists $\eta > 0$ such that if $|x| < 2c, |y| < 2c$, and $|x-y| \leq \eta$, then $|\psi(x)-\psi(y)| \leq \varepsilon$. We assume without loss of generality that $\eta < c$.

Finally, as $(X_t - X_0)$ is a Lévy process, by definition it converge in probability to 0 when $t \to 0$, so that there exists $t_0 > 0$ such that for every $0 \leq t \leq t_0$, $\mathbb{P}\{|X_t - X_0| > \eta\} \leq \varepsilon$. Thus we have:

$$\begin{aligned}\mathbb{E}\left[|\psi(X_t) - \psi(X_0)|^2\right] &= \mathbb{E}\left[|\psi(X_t) - \psi(X_0)|^2 \mathbb{1}_{\{|X_t - X_0| > \eta\}}\right] \\ &\quad + \mathbb{E}\left[|\psi(X_t) - \psi(X_0)|^2 \mathbb{1}_{\{|X_t - X_0| \leq \eta,\ |X_0| < c\}}\right] \\ &\quad + \mathbb{E}\left[|\psi(X_t) - \psi(X_0)|^2 \mathbb{1}_{\{|X_t - X_0| \leq \eta,\ |X_0| > c\}}\right] \\ &\leq 2\,\|\psi\|_\infty \mathbb{P}\{|X_t - X_0| > \eta\} + \varepsilon^2 + 4\varepsilon^2 \\ &\leq 2\,\|\psi\|_\infty \varepsilon + 5\varepsilon^2.\end{aligned}$$

This gives the result. □

IV Classical evolution of an observable with discrete spectrum

In the previous section, we considered CP maps in the Heisenberg picture, that is acting on observables. In this section, we enlarge the study to their predual.

Definition IV.1. *The* predual *of a completely positive map $\mathcal{L}$ is the CP map $\mathcal{L}_*$ on $\mathcal{L}_1(\mathcal{H})$ defined by*

$$Tr[\mathcal{L}_*(\rho)X] = Tr[\rho\mathcal{L}(X)],$$

for every $X \in \mathcal{B}(\mathcal{H})$ and $\rho \in \mathcal{L}_1(\mathcal{H})$.

We focus on the case where E is discrete, either finite or infinite, and we note $E = \{1, ..., N\}$, where $N = \dim \mathcal{H} \in \mathbb{N} \cup \{+\infty\}$. Consequently $\mathcal{A}$ and its predual algebra $\mathcal{A}_*$ can be written:

$$\mathcal{A} = \left\{ \sum_{i \geq 0} f(i) |i\rangle\langle i|, \ f \in L^\infty(E) \right\},$$
$$\mathcal{A}_* = \left\{ \sum_{i \geq 0} g(i) |i\rangle\langle i|, \ g \in L^1(E) \right\}.$$

In this setup, as writen above, the predual $\mathcal{A}_*$ is a subspace of all trace-class operators $\mathcal{L}_1(\mathcal{H})$. A natural question is whether the predual of a subclassical CP map $\mathcal{L}$ on $\mathcal{H}$ admits $\mathcal{A}_*$ as a stable commutative subalgebra or not. We will mainly be concerned with discrete time dynamics. Let $\mathcal{L}$ be a $\mathcal{A}$-subclassical CP map. As it was already pointed out in Section III. 4, its classical restriction is entirely described by a stochastic matrix Q on E, such that

$$Q_{x_1,x_2} = \mathrm{Tr}[|x_1\rangle\langle x_1|\mathcal{L}(|x_2\rangle\langle x_2|)]. \tag{IV.1}$$

At this point, a natural question is wether the CP map $\mathcal{L}_*$ is itself $\mathcal{A}_*$-subclassical, and in this case what is its restriction. In Subsection IV. 1 we propose a classification of $\mathcal{A}$-subclassical CP maps based on this question. In Subsection IV. 2 we give examples of each class of CP map we defined, showing how they appear naturally in quantum physics. Subsection IV. 3 is devoted to a theorem which makes the link between general $\mathcal{A}$-subclassical CP maps and the ones emerging as extension of dynamical systems.

IV. 1 Classification of subclassical dynamics with discrete spectrum

We start this section with the following remark. Take $f \in L^\infty(E)$ and $g \in L^1(E)$. Then we have

$$\begin{aligned}
\mathrm{Tr}\{\mathcal{L}_*[M_g]M_f\} &= \mathrm{Tr}\{M_g\mathcal{L}[M_f]\} \\
&= \mathrm{Tr}\{M_g M_{Qf}\} \\
&= \sum_{i=1}^{N} g(i)\, Qf(i) \\
&= \sum_{i=1}^{N} gQ(i)\, f(i) \\
&= \mathrm{Tr}\{M_{gQ}M_f\}.
\end{aligned}$$

This is however not enough to claim that $\mathcal{L}_*[M_g] = M_{gQ}$, as we did not test it over all $\mathcal{B}(\mathcal{H})$ but only over $\mathcal{A}$. As we will see, there are indeed subclassical CP maps which do not have this property.

A trace-preserving CP map $\mathcal{L}_*$ acting on $\mathcal{L}_1(\mathcal{H})$ such that $\mathcal{A}_*$ is a stable subalgebra will also be called a *$\mathcal{A}_*$-subclassical CP map.* Thus, not all predual evolutions of $\mathcal{A}$-subclassical dynamics are themselves $\mathcal{A}$-subclassical. This is the starting remark for the classification we propose here.

Definition IV.2. *Let $\mathcal{L}$ be a normal identity preserving CP map on $\mathcal{B}(\mathcal{H})$. We say that :*

- *$\mathcal{L}$ is a* doubly $\mathcal{A}$-subclassical completely positive map *if $\mathcal{L}$ is $\mathcal{A}$-subclassical and $\mathcal{L}_*$ is $\mathcal{A}_*$-subclassical (the definition naturally expands to QMS);*
- *$\mathcal{L}$ is a* measurement $\mathcal{A}$-subclassical completely positive map *if $\mathcal{L}(X) \in \mathcal{A}$ for all $X \in \mathcal{B}(\mathcal{H})$;*
- *$\mathcal{L}_*$ is a* measurement $\mathcal{A}_*$-subclassical completely positive map *if $\mathcal{L}_*(\rho) \in \mathcal{A}_*$ for all $\rho \in \mathcal{L}_1(\mathcal{H})$;*
- *$\mathcal{L}$ is a* purely $\mathcal{A}$-subclassical completely positive map *if $\mathcal{L}$ is both a doubly and a measurement $\mathcal{A}$-subclassical CP map.*

Note that we can not define $\mathcal{A}$-subclassical QMS in continuous time, for the reason that $\mathcal{P}_0(X) = X$ for all $X \in \mathcal{B}(\mathcal{H})$. Thus there does not exist QMS $(\mathcal{P}_t)_{t\geq 0}$ such that for all $X \in \mathcal{B}(\mathcal{H})$, $\mathcal{P}_t(X) \in \mathcal{A}$ for all $t \geq 0$.
Those definitions are linked with von Neumann description of a measurement on a quantum system.

Definition IV.3. *The* von Neumann measurement operator *of the observable A on $\mathcal{H}$ is the operator $\mathcal{M}$ on $\mathcal{B}(\mathcal{H})$ defined for all $X \in \mathcal{B}(\mathcal{H})$ by*

$$\mathcal{M}(X) = \sum_{k=1}^{N} < k|X|k > |k\rangle\langle k|, \qquad \text{(IV.2)}$$

where the sum converges in the strong sense whenever $N = +\infty$.

Physically, the von Neumann measurement operator model the system put in a measuring device, without the experimentator knowing the result of the measurement. Denote by $\mathcal{A}_{\text{off}}$ the closed (in norm, strong and weak* topologies) subspace of $X \in \mathcal{B}(\mathcal{H})$ such that $\langle k|X|k\rangle = 0$ for all $k \in V$.

Lemma IV.1. *$\mathcal{M}$ is a norm one projection on $\mathcal{A}$, with kernel $\mathcal{A}_{off}$.*

Proof. The fact that $\mathcal{M}$ is a projection on $\mathcal{A}$ with kernel $\mathcal{A}_{\text{off}}$ is an easy computation. Consequently we have $\|\mathcal{M}\| \geq 1$. The fact that $\|\mathcal{M}\| = 1$ is a well-known result on CP maps, since $\|\mathcal{M}\| = \|\mathcal{M}(I)\| = \|I\| = 1$. □

We have equivalent definitions of the four kind of CP maps, related to $\mathcal{M}$ and $\mathcal{A}_{\text{off}}$.

Proposition IV.1.

(i) A CP map $\mathcal{L}$ is doubly $\mathcal{A}$-subclassical if and only if $\mathcal{A}_{off}$ is stable under its action.

(ii) A CP map $\mathcal{L}$ on $\mathcal{B}(\mathcal{H})$ is a measurement $\mathcal{A}$-subclassical CP map if and only if $\mathcal{M} \circ \mathcal{L} = \mathcal{L}$.

(iii) A CP map $\mathcal{L}_*$ on $\mathcal{L}_1(\mathcal{H})$ is a measurement $\mathcal{A}_*$-subclassical CP map if and only if $\mathcal{M} \circ \mathcal{L}_* = \mathcal{L}_*$.

(iv) A $\mathcal{A}$-subclassical CP map $\mathcal{L}$ is purely $\mathcal{A}$-subclassical if and only if $\mathcal{L}$ vanish on $\mathcal{A}_{off}$.

Proof. Assertions (ii) and (iii) are straightforward. We prove (i). By duality, we just have to prove that $\mathcal{A}_*^{\perp} = \mathcal{A}_{\text{off}}$, where $\mathcal{A}_*^{\perp} = \{X \in \mathcal{B}(\mathcal{H}),\ \text{Tr}[\rho X] = 0 \text{ for all } \rho \in \mathcal{A}_*\}$. But $\mathcal{A}_*$ is the norm closure of the space generated by the orthogonal projectors $|k\rangle\langle k|$ for all $k \in V$. Thus we have

$$\begin{aligned} X \in \mathcal{A}_*^{\perp} &\Leftrightarrow \text{Tr}[|k\rangle\langle k|X] = 0 \text{ for all } k \in V; \\ &\Leftrightarrow \langle k|X|k\rangle = 0 \text{ for all } k \in V; \\ &\Leftrightarrow X \in \mathcal{A}_{\text{off}}, \end{aligned}$$

which proves the result. Let's prove (iv). A direct corollary of Lemma IV.1 is the fact that $\mathcal{A}$ and $\mathcal{A}_{\text{off}}$ are closed complemented subspaces, i.e.

$$\mathcal{A} \bigoplus \mathcal{A}_{\text{off}} = \mathcal{B}(\mathcal{H}). \tag{IV.3}$$

If $\mathcal{L}(\mathcal{A}_{\text{off}}) = \{0\}$, we just have to prove that $\mathcal{L}(\mathcal{B}(\mathcal{H})) \subset \mathcal{A}$. Yet, if $X \in \mathcal{B}(\mathcal{H})$ we can write $X = \mathcal{M}(X) + X - \mathcal{M}(X)$, with $\mathcal{M}(X) \in \mathcal{A}$ and $X - \mathcal{M}(X) \in \mathcal{A}_{\text{off}}$, so our assumption and the fact that $\mathcal{L}$ is $\mathcal{A}$-subclassical allow us to conclude.
In the other direction, if $\mathcal{L}$ is purely $\mathcal{A}$-subclassical, we have $\mathcal{L}(\mathcal{A}_{\text{off}}) \subset \mathcal{A} \cap \mathcal{A}_{\text{off}} = \{0\}$. □

Note that a purely $\mathcal{A}$-subclassical CP map is entirely defined by its restriction to $\mathcal{A}$. It is itself a classical Markov operator; this justifies the name we give to such CP map. Consequently, the purely $\mathcal{A}$-subclassical CP map whose restriction on $\mathcal{A}$ is a given stochastic matrix Q is unique.

In order to see why those definitions are natural, let see what happens in finite dimension. In this case, we can endow $\mathcal{B}(\mathcal{H})$ with the structure of an Hilbert space, by taking the Hilbert-Schmidt scalar product $(\cdot,\cdot)_{\mathrm{HR}}$:

$$(X,Y)_{\mathrm{HR}} = \mathrm{Tr}[X^*Y].$$

For this scalar product, the family of operators $(|i\rangle\langle j|)_{i,j\in V}$ is an orthonormal basis of $\mathcal{B}(\mathcal{H})$. In this basis, a $\mathcal{A}$-subclassical CP map $\mathcal{L}$ can be written

$$\mathcal{L} = \begin{pmatrix} Q & B \\ 0 & C \end{pmatrix}, \tag{IV.4}$$

where Q is the restriction to $\mathcal{A}$ of $\mathcal{L}$, i.e. a stochastic matrix, B an $N(N-1)\times N$ matrix and C an operator on $\mathcal{A}_{off}$, i.e. an $N(N-1)$-square matrix. The following proposition is straightforward:

Proposition IV.2. *Let $\mathcal{L}$ be a $\mathcal{A}$-subclassical CP map. we write $\mathcal{L}$ as in equation (IV.4). Then:*

1) Q is a stochastic matrix on $\mathbb{C}^N$;

2) $\mathcal{L}$ is a doubly $\mathcal{A}$-subclassical CP map iff $B = 0$;

3) $\mathcal{L}$ is a measurement $\mathcal{A}$-subclassical CP map iff $C = 0$;

4) $\mathcal{L}$ is a purely $\mathcal{A}$-subclassical CP map iff both $B = 0$ and $C = 0$.

We now turn to our first examples.

IV. 2 First examples

We begin with an example of measurement $\mathcal{A}$-subclassical CP map.

Proposition IV.3. *Let U be a unitary operator on $\mathcal{H}$. Then the CP map $\mathcal{L}$ defined for all $X \in \mathcal{B}(\mathcal{H})$ by*

$$\mathcal{L}(X) = \mathcal{M}[U^*XU], \tag{IV.5}$$

is a measurement $\mathcal{A}$-subclassical CP map, whose classical restriction to $\mathcal{A}$ is given by the stochastic matrix $Q = (Q_{i,j})_{i,j\in V}$

$$Q_{i,j} = |\langle j|U|i\rangle|^2 \tag{IV.6}$$

Proof. The fact that $\mathcal{L}$ is a measurement subclassical CP map is obvious with Proposition IV.3. Equation (IV.6) is an easy computation that we leave to the reader. □

The previous proposition inspires a larger class of measurement subclassical CP map. Let Q be a stochastic matrix, either in finite or infinite dimension. We write for all $k \in V$

$$M_k = \sum_{l\in E} \sqrt{Q_{k,l}}\ |l\rangle\langle k|. \tag{IV.7}$$

Note that for every $i, j, k = 1, ..., n$, we have $M_k^* |i\rangle\langle j| M_k = \sqrt{Q_{k,i}Q_{k,j}} |k\rangle\langle k|$, which we could interpret as the fact that the channel $M_j^* \cdot M_j$ project onto the space generated by $|j\rangle\langle j|$. The measurement subclassicality follows easily, when we write, for $X \in \mathcal{B}(\mathcal{H})$, $X = \sum_{i,j\in E} \langle i|X|j\rangle |i\rangle\langle j|$, where the convergence of the sum is in the strong sense.

The examples of doubly subclassical and purely subclassical dynamics we are going to give in this section both come from the same physical concept, called weak-coupling or Van Hoove limit. The book [AK02] contains a lot of examples of such QMS. The starting point is the following theorem (see [AFH06]):

Proposition IV.4. *Let Q be a stochastic matrix on V (possibly infinite if V is). Then the linear map $\Phi[Q] : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ defined by*

$$\Phi[Q](X) = \sum_{i,j=1}^{N} Q_{i,j} \langle j|X|j\rangle |i\rangle\langle i|, \tag{IV.8}$$

is a purely completely positive map associated to the stochastic matrix Q.

Thus for any stochastic matrix Q on E, the map $\Phi(Q)$ is the only purely $\mathcal{A}$-subclassical CP map whose restriction to $\mathcal{A}$ is Q.

From a purely $\mathcal{A}$-subclassical CP map we can associate a doubly $\mathcal{A}$-subclassical QMS called *generic QMS*. To do that, we note M_d the diagonal part of a matrix M on E, and $\sqrt{M_d} = diag(\sqrt{M_{1,1}}, ..., \sqrt{M_{n,n}})$. Take a classical Markov semigoup $(P_t)_{t\geq 0}$ on $L^\infty(E)$, with generator B.

Proposition IV.5. *The process defined for all time $t \geq 0$ and for all $X \in \mathcal{B}(\mathcal{H})$ by*

$$\mathcal{P}_t(X) = \Phi[P_t](X) - \Phi[e^{tB_d}](X) + \sqrt{B_d} X \sqrt{B_d}, \tag{IV.9}$$

is a doubly $\mathcal{A}$-subclassical QMS, whose restriction to $\mathcal{A}$ is the Markov semigroup $(\mathcal{P}_t)_{t\geq 0}$.

Proof. See [AFH06] □

IV. 3 Subclassical CP map and quantum trajectories

In Section III. 3, we constructed quantum extensions of reversible dynamical systems for discrete state spaces, leading to subclassical CP maps when taking the trace over the environment. Those CP maps had the particularity to be the trace of the reversible dynamical system. In this special case we remarked that reversibility of the dynamical system were a necessary condition for the existence of a unitary extension, a fact which is not true in general. In this Subsection we enlarge the study to general subclassical CP maps. Theorem IV.1 below states a relation between the classical restriction and the unitary operator giving the dilation.

Let $\mathcal{L}$ be a $\mathcal{A}$-subclassical CP map. Using a certain form of Stinespring Theorem, we know there exist an Hilbert space $\mathcal{K} = l^2(F)$ with $F = \{0, \ldots, M\}$ and $M \in \mathbb{N} \cup \{+\infty\}$, a state ω on $\mathcal{K}$ and a unitary operator U on $\mathcal{H} \otimes \mathcal{K}$ such that

$$\mathcal{L}(X) = \mathrm{Tr}_\omega[U^*(X \otimes I_\mathcal{K})U].$$

Let $(|y\rangle)_{y\in F}$ be an orthonormal basis of $\mathcal{K}$ that diagonalize ω, and define the probability measure ν_ω on F, by $\nu_\omega(y) = \langle y|\omega|y\rangle$, so that

$$\omega = \sum_{y\in F} \nu_\omega(y)|y\rangle\langle y|.$$

Finally we write $\mathcal{B}$ the commutative algebra generated by $B = \sum_{y\in F} y\ |y\rangle\langle y|$.

Of course, $X \mapsto U^* X U$ is not $\mathcal{A} \otimes \mathcal{B}$-subclassical in general. However, given U, it is easy to characterize when it is the case.

Theorem IV.1. *Define on $E \times F$ the stochastic matrix R by:*

$$R_{(x_1,y_1),(x_2,y_2)} = |\langle x_2, y_2|\ U\ |x_1, y_1\rangle|^2 . \tag{IV.10}$$

Then R is deterministic if and only if the commutative subalgebra $\mathcal{A}\otimes\mathcal{B}$ is stable under the action of $U^*\cdot U$.
In this case, $\mathcal{L}$ is doubly $\mathcal{A}$-subclassical.

Moreover, let Q be the stochastic matrix on E which is the restriction of $\mathcal{L}$ on $\mathcal{A}$. Then for every $x_1, x_2 \in E$ we have

$$Q_{x_1,x_2} = \sum_{y_1,y_2\in F} \nu_\omega(y_1) R_{(x_1,y_1),(x_2,y_2)}. \tag{IV.11}$$

Thus Q is the trace over (F,ν_ω) of the stochastic matrix R.

Proof. We begin by the proof of the first part of the theorem. If $\mathcal{A}\otimes\mathcal{B}$ is stable under the action of the unitary evolution $X\in\mathcal{B}(\mathcal{H}\otimes\mathcal{K})\mapsto U^*XU$, then by Proposition III.1 the map R is determinist, i.e. it is a permutation matrix on $E\times F$.
Conversely, suppose that R is a determinist: there exists an application $T: E\times F\to E\times F$ such that

$$R_{(x,y),(x',y')} = \delta^{(x',y')}_{T(x,y)}.$$

As U is unitary, it is clear that T is invertible. A direct computation using Equation (IV.10) shows that for all $(x,y)\in E\times F$, we have $U|x,y\rangle\langle x,y|U^* = |T^{-1}(x,y)\rangle\langle T^{-1}(x,y)|$. It is itself equivalent to equation to $U^*|x,y\rangle\langle x,y|U = |T(x,y)\rangle\langle T(x,y)|$, so that $\mathcal{A}\otimes\mathcal{B}$ is stable under the action of the unitary evolution.
Let prove that in this case $\mathcal{L}_*$ is $\mathcal{A}_*$-subclassical. We have for all $\rho\in\mathcal{L}_1(\mathcal{H})$:

$$\mathcal{L}_*(\rho) = \mathrm{Tr}_{\mathcal{K}}[U(\rho\otimes\omega)U^*].$$

We write $X:(x,y)\in E\times F\mapsto X(x,y)$ the coordinate map of T^{-1} over E. Then for all $x\in E$, we have

$$\begin{aligned}\mathcal{L}_*(|x\rangle\langle x|) &= \mathrm{Tr}_{\mathcal{K}}[U(|x\rangle\langle x|\otimes\omega)U^*]\\ &= \sum_{y\in F}\mu_\omega(y)\mathrm{Tr}_{\mathcal{K}}[|T^{-1}(x,y)\rangle\langle T^{-1}(x,y)|]\\ &= \sum_{y\in F}\mu_\omega(y)|X(x,y)\rangle\langle X(x,y)|,\end{aligned}$$

and the result follows by linearity. Now we prove Equation (IV.11). For all $x_1, x_2\in E$, we have

$$\begin{aligned} Q_{x_1,x_2} &= \mathrm{Tr}[|x_1\rangle\langle x_1|\mathcal{L}(|x_2\rangle\langle x_2|)] \\ &= \mathrm{Tr}[|x_1\rangle\langle x_1|\mathrm{Tr}_\omega[U^*(|x_2\rangle\langle x_2|\otimes I)U^*] \\ &= \mathrm{Tr}\left[|x_1\rangle\langle x_1|\otimes\omega\left\{U(|x_2\rangle\langle x_2|\otimes I)U^*\right\}\right] \\ &= \sum_{y\in F}\langle x_1,y|U^*(|x_2\rangle\langle x_2|\otimes I)U|x_1,y\rangle\nu_\omega(y). \end{aligned}$$

To end the proof we just have to check that

$$\langle x_1,y|U^*(|x_2\rangle\langle x_2|\otimes I_{\mathcal{K}})U|x_1,y\rangle = \sum_{y'\in F}|u_{(x_2,y'),(x_1,y)}|^2,$$

which is a straightforward computation. □

Remarks IV.1. This interpretation of the stochastic matrix R only holds for one step of the evolution. Indeed, Equation (IV.11) does not hold in full generality for Q^2 and R^2. The reason is that R^2 does not satisfy Equation (IV.10) with U^2 instead of U.

The stochastic matrix R defined by Equation IV.10 also appears in the context of discrete time quantum trajectories (see [Pel10b]).

Let U be a unitary operator on $\mathcal{H}\otimes\mathcal{K}$, with no further hypothesis. In the Schrödinger picture, the evolution of states on $\mathcal{H}$ is given by the CP map $\mathcal{L}_*$ defined by

$$\mathcal{L}_*(\rho) = \mathrm{Tr}_{\mathcal{K}}[U\rho\otimes\omega U^*].$$

Suppose now that $\mathcal{H}$ is in the state $\rho_0 = \sum_{x\in E}\mu(x)|x\rangle\langle x|$, where μ is a probability measure on E and that $\mathcal{K}$ is in the pure state $\omega = |0\rangle\langle 0|$. In order to observe the evolution of ρ_0, without destroying it, we make a measurement on the environment $\mathcal{K}$. Decompose U in the basis $(|y>)_{y\in F}$:

$$U = \sum_{y,y'\in F} U_y^{y'}\otimes|y\rangle\langle y'|.$$

Then we can write the corresponding Kraus decomposition for $\mathcal{L}_*$:

$$\mathcal{L}_*(\cdot) = \sum_{y\in F} M_y\cdot M_y^*, \tag{IV.12}$$

where $M_y = U_y^0$. Now if we make a measurement along the basis $(|y\rangle)_{y\in F}$ after the interaction, we obtain on $\mathcal{H}$ the random state

$$\rho_1(y) = \frac{M_y\rho_0M_y^*}{\mathrm{Tr}[M_y\rho_0M_y^*]} \text{ with probability } p(y) = \mathrm{Tr}[M_y\rho_0M_y^*]. \tag{IV.13}$$

The transformation $\rho_0 \mapsto \rho_1(y)$ is thus a state-value measure on F, with law given by the probability measure $p = (p(0), ..., p(M))$. This law is directly obtain via the operator R:

Proposition IV.6. *Define the stochastic matrix R on $E \times F$ by Equation (IV.10). Then the probability measure $p = (p(0), ..., p(M))$ on F defined by Equation (IV.13) is the marginal over F of the law $[(\mu \otimes \delta_0)R]$ on $E \times F$, i.e for all $y \in F$:*

$$p(y) = [(\mu \otimes \delta_0)R](E, y). \tag{IV.14}$$

Proof. We have

$$\begin{aligned}
[(\mu \otimes \delta_0)R](E, y) &= \sum_{x,x' \in E} \mu(x) R_{(x,0),(x',y)} \\
&= \sum_{x,x' \in E} \mu(x) \left|< x', y|U|x, 0 >\right|^2 \\
&= \sum_{x,x' \in E} \mu(x) < x', y|U|x, 0 >< x, 0|U^*|x', y > \\
&= \sum_{x \in E} \mu(x) \mathrm{Tr}\left[\left(\sum_{x' \in E} |x', y >< x', y|\right) U|x, 0 >< x, 0|U^*\right] \\
&= \mathrm{Tr}\left[I_{\mathcal{H}} \otimes |y >< y| \; U \sum_{x \in E} \mu(x)|x >< x| \otimes \omega U^*\right] \\
&= \mathrm{Tr}\left\{\mathrm{Tr}_{\mathcal{K}}\left[I_{\mathcal{H}} \otimes |y >< y| \; U\rho_0 \otimes \omega U^*\right]\right\} \\
&= \mathrm{Tr}[M_y \rho_0 M_y^*] \\
&= p(y).
\end{aligned}$$

□

Remarks IV.2. As for Remark IV.1, this interpretation of the matrix R only holds for one step of the evolution.

IV. 4 Physical examples

The stochastic matrix R in Theorem IV.1 is always doubly stochastic, because U is a unitary operator. Using Birkhoff-von Neumann Theorem, we can decompose R as a convex combination of permutation matrices: theres exist $V_1, ...V_p$ permutation matrices on $E \times F$ and $\lambda_1, ..., \lambda_p$ such that

$$R = \sum_{l=1}^{p} \lambda_l V_l. \tag{IV.15}$$

Consequently, for each $l = 1, \dots, p$ we can associate an invertible application $T_l : E \times F \to E \times F$. Then Equation IV.11 can be written:

$$Q_{x_1,x_2} = \sum_{l=1}^{p} \lambda_l \left[\sum_{y_1,y_2 \in F} \delta_{T(x_1,y_1)}^{(x_2,y_2)} \nu_\omega(y_1) \right].$$

Thus the classical evolution given by Q is always the trace of a mixture of dynamical systems on a bigger system. In order to illustrate this remark and Theorem IV.1, we now give two physical examples of subclassical CP maps. We take $\mathcal{H} = \mathcal{K} = \mathbb{C}^2$, so that $E = F = \{0,1\}$. We will describe the interaction between the two systems by expliciting their Hamiltonian H in the orthonormal basis $(|i\rangle \otimes |j\rangle)_{i,j=0,1}$ of $\mathcal{H} \otimes \mathcal{K}$.

Examples IV.1 (Spontaneous emission)**.** For this example, $\mathcal{H}$ and $\mathcal{K}$ are two-levels atoms, so that $|0\rangle$ corresponds to the ground state while $|1\rangle$ corresponds to the excited state for both the system and the environment. During the time $t > 0$ both systems interact with each other, they evolve in the following way:

- if the states of the two systems are the same (both fundamental or both excited), then nothing happens;
- if they are different (one fundamental and the other excited), then they can either exhanged their energies or stay as they are, depending on a parameter $\theta \in \mathbb{R}$.

The Hamiltonian is then

$$H = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i\theta & 0 \\ 0 & i\theta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{IV.16}$$

and the unitary operator $U = e^{-itH}$:

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(t\theta) & -\sin(t\theta) & 0 \\ 0 & \sin(t\theta) & \cos(t\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{IV.17}$$

We can compute the stochastic matrix R given by Theorem IV.1 and its decomposition as permutation matrices:

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos^2(t\theta) & \sin^2(t\theta) & 0 \\ 0 & \sin^2(t\theta) & \cos^2(t\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \cos^2(t\theta) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \sin^2(t\theta) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{IV.18}$$

This decomposition is what we expected: when they have different energy levels, both systems stay as they are with a probability $\cos^2(t\alpha)$, or they exchange their energy with probability $\sin^2(t\alpha)$.

Examples IV.2 (Spin system)**.** For this example, $\mathcal{H}$ and $\mathcal{K}$ modeled the spin of two atoms. In the coordinates x, y, z, $|0\rangle$ corresponds to the spin up in the axe Oz while $|1\rangle$ corresponds to the spin down. In this basis, the Pauli matrices are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{IV.19}$$

We consider the following interaction between the system and the environment (λ and μ are two real numbers):

$$H = \lambda\sigma_x \otimes \sigma_x + \mu\sigma_y \otimes \sigma_y, \tag{IV.20}$$

and the unitary operator $U = e^{-itH}$:

$$U = \begin{pmatrix} -\cos t(\lambda-\mu) & 0 & 0 & -i\sin t(\lambda-\mu) \\ 0 & -\cos t(\lambda+\mu) & i\sin t(\lambda+\mu) & 0 \\ 0 & i\sin t(\lambda+\mu) & -\cos t(\lambda+\mu) & 0 \\ i\sin t(\lambda-\mu) & 0 & 0 & -\cos t(\lambda-\mu) \end{pmatrix}. \tag{IV.21}$$

Computing the stochastic matrix R leads to

$$R = \begin{pmatrix} \cos^2 t(\lambda-\mu) & 0 & 0 & \sin^2 t(\lambda-\mu) \\ 0 & \cos^2 t(\lambda+\mu) & \sin^2 t(\lambda+\mu) & 0 \\ 0 & \sin^2 t(\lambda+\mu) & \cos^2 t(\lambda+\mu) & 0 \\ \sin^2 t(\lambda-\mu) & 0 & 0 & \cos^2 t(\lambda-\mu) \end{pmatrix}, \tag{IV.22}$$

which can be decomposed as

$$R = a\begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix} + b\begin{pmatrix} 1&0&0&0 \\ 0&0&1&0 \\ 0&1&0&0 \\ 0&0&0&1 \end{pmatrix} + c\begin{pmatrix} 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \\ 1&0&0&0 \end{pmatrix} + d\begin{pmatrix} 0&0&0&1 \\ 0&1&0&0 \\ 0&0&1&0 \\ 1&0&0&0 \end{pmatrix}, \tag{IV.23}$$

with

$$a = \cos^2 t(\lambda-\mu)\cos^2 t(\lambda+\mu), \ b = \cos^2 t(\lambda-\mu)\sin^2 t(\lambda+\mu),$$
$$c = \sin^2 t(\lambda-\mu)\sin^2 t(\lambda+\mu), \ d = \cos^2 t(\lambda-\mu)\sin^2 t(\lambda-\mu).$$

Chapter 4

Open Quantum Walk

Introduction

Open Quantum Walks (OQW) are very typical completely positive maps (CP maps) which display classical behavior due to the invariance of a certain commutative algebra. They were originally designed to model quantum walks which take dissipative effect into account [APSS12]. They are also powerful tools to obtain tractable models of out-of-equilibrium systems. We will come back latter on their definition.

Due to the classical part of the CP map, there is an available set of vertices that allows to consider geometric aspect in the study of such CP maps. In particular, it is possible to define a Dirichlet problem associated to a subdomain of this set.

First let recall the definition of the discrete Dirichlet problem. Let V be a set of vertices, possibly infinite, and let D be a strict subdomain of V. Let Q be a stochastic matrix indexed by V, that models the out-of-equilibrium dynamics of a particle constrained in moving on V. The Dirichlet problem question the existence of a solution f to the following equation:

$$\begin{aligned} (I-Q)f(i) &= a(i) \quad \forall i \in \mathrm{int}\, D, \\ f(j) &= b(j) \quad \forall j \in \partial D, \end{aligned}$$

where $\mathrm{int}\, D$ denote the interior of D and ∂D its boarder (whose precise definition we shall give latter), where a, b and f are bounded complex-value functions on V. The function a is called the *inner* data while the function b is called the *outer* data.

There are two usual approaches to deal with this problem. The first one is probabilistic in nature and requires to compute exit-times associated to the domain D.

The second one is based on the theory of Dirichlet forms and uses Hilbertian techniques to express the solution f as a minimizer of some functional.

In the case of OQW, we still consider the sets V and D but to each vertex $i \in V$ we associate an Hilbert space $\mathfrak{h}_i$ that is thought as modeling the internal degrees of freedom of a particle. The quantum state-space is thus $\mathcal{H} = \bigoplus_{i \in V} \mathfrak{h}_i$. Then the inner and outer data are observables in $\bigoplus_{i \in V} \mathcal{B}(\mathfrak{h}_i)$.

In the article in preparation with Denis Bernard and Yan Pautrat, we address both approaches mentioned above in order to solve the Dirichlet problem. My own contribution consisting in the second one, it is the one we shall develop in this chapter.

This chapter is structured as follows. In section I we recall the definition of OQW. In section II we define the non commutative Dirichlet form associated to an OQW and prove the equivalence between the Dirichlet form and the OQW. In section III we state our main result on the Dirichlet problem.

I Open quantum walks and notations

We start this section with a short presentation of open quantum walks. For more details we refer the reader to [APSS12].

We consider a Hilbert space $\mathcal{H}$ of the form $\mathcal{H} = \bigoplus_{i \in V} \mathfrak{h}_i$ where V is a countable set of vertices and each $\mathfrak{h}_i$ is a separable Hilbert space. We view $\mathcal{H}$ as describing the degrees of freedom of a particle constrained to move on V: the "V-component" describes the spatial degrees of freedom (the position of the particle) while $\mathfrak{h}_i$ describes the internal degrees of freedom of the particle, when it is located at site $i \in V$.

For book-keeping purposes we denote the subspace $\mathfrak{h}_i$ of $\mathcal{H}$ by $\mathfrak{h}_i \otimes |i\rangle$. Therefore, whenever a vector $\varphi \in \mathcal{H}$ belongs to the subspace $\mathfrak{h}_i$, we will denote it by $\varphi \otimes |i\rangle$ and drop the (implicit) assumption that $\varphi \in \mathfrak{h}_i$. Similarly, when an operator A on $\mathcal{H}$ satisfies $\mathfrak{h}_j^\perp \subset \operatorname{Ker} A$ and $\operatorname{Ran} A \subset \mathfrak{h}_i$, we denote it by $A = L_{i,j} \otimes |i\rangle\langle j|$ where $L_{i,j}$ is viewed as an operator from $\mathfrak{h}_j$ to $\mathfrak{h}_i$. This will allow us to use the same notation as in e.g. [APSS12, AGPS15, KY13, LS14, Pel14].

Definition I.1. *An* Open Quantum Walk *on $\mathcal{H}$ is a completely positive map on $\mathcal{L}_1(\mathcal{H})$ of the following form:*

$$\mathfrak{M} : \rho \mapsto \sum_{i,j \in V} \tilde{T}_{i,j}(\rho) \tag{I.1}$$

where, for any i, j in V, $\tilde{T}_{i,j}$ is a map of the form $T_{i,j} \otimes |i\rangle\langle j| \cdot |j\rangle\langle i|$, where $T_{i,j}$ is a CP map from $\mathcal{L}_1(\mathfrak{h}_j)$ to $\mathcal{L}_1(\mathfrak{h}_i)$ such that

$$\forall j \in V \quad \sum_{i \in V} T_{i,j}^*(I_{\mathfrak{h}_i}) = I_{\mathfrak{h}_j}, \tag{I.2}$$

where the series is meant in the strong convergence sense if V is infinite.

The $T_{i,j}$ are thought of as encoding both the probability of a transition from site j to site i and the effect of that transition on the internal degrees of freedom. Equation (I.2) therefore encodes the "stochasticity" of the transitions $T_{i,j}$ and immediately implies that $\operatorname{Tr} \mathfrak{M}(\rho) = \operatorname{Tr} \rho$ for any $\rho \in \mathcal{L}_1(\mathcal{H})$.

Remark I.1. This definition differs from the original one of Attal and al. in [APSS12], where they additionally ask that the $T_{i,j}$'s are of the form

$$\rho \otimes |j\rangle\langle j| \mapsto L_{i,j} \rho L_{i,j}^* \otimes |i\rangle\langle i|,$$

for some bounded operators $L_{i,j}$ from $\mathfrak{h}_j$ to $\mathfrak{h}_i$. Our generalization consists in taking the convex hull of such CP maps.

A crucial remark is that the image of $\mathfrak{M}$ is a subset of the class of "diagonal" states, i.e. states of the form

$$\sum_{i \in V} \rho(i) \otimes |i\rangle\langle i|. \tag{I.3}$$

In addition, even if ρ is not diagonal, then $\mathfrak{M}(\rho)$ depends only on its diagonal elements $\rho(i,i)$ in the decomposition $\rho = \sum_{i,j \in V} \rho(i,j) \otimes |i\rangle\langle j|$. Therefore, from now on, we will only consider states of the form (I.3). The action of $\mathfrak{M}$ on such states takes the form

$$\mathfrak{M}(\rho) = \sum_{i \in V} \Big\{ \sum_{j \in V} T_{i,j}(\rho(j)) \Big\} \otimes |i\rangle\langle i|. \tag{I.4}$$

This implies the following Lemma.

Lemma I.1. *Let P_i denote the orthogonal projection on $\mathfrak{h}_i$ and write $\mathcal{A}$ the map:*

$$\begin{array}{rccc} \mathcal{A}: & \mathcal{B}(\mathcal{H}) & \to & \mathcal{B}(\mathcal{H}) \\ & X & \mapsto & \sum_{i \in V} P_i X P_i \end{array} \tag{I.5}$$

Remark that $\mathcal{A}$ is the conditional expectation on its image, in the sense of von Neumann algebra theory. Then a quantum channel[1] $\mathfrak{M}$ on $\mathcal{B}(\mathcal{H})$ is an OQW if and only if

$$\mathcal{A} \circ \mathfrak{M} = \mathfrak{M} \circ \mathcal{A} = \mathfrak{M} \tag{I.6}$$

[1] We call quantum channel a trace-preserving CP map acting on the trace-class operators

Remark that, in particular, an invariant state π necessarily has the form

$$\pi = \sum_{i\in V} \pi(i) \otimes |i\rangle\langle i|.$$

We now describe the (classical) processes of interest, associated with $\mathfrak{M}$. We let $\Omega = V^{\mathbb{N}}$ and for any state ρ on $\mathcal{H}$ of the form (I.3), we define on Ω a probability by defining its restrictions to V^{n+1} for $n \geq 0$:

$$\mathbb{P}_\rho(i_0, \dots, i_n) = \mathrm{Tr}\big[T_{i_n,i_{n-1}} \circ \cdots \circ T_{i_1,i_0}(\rho(i_0))\big]. \tag{I.7}$$

Relation (I.2) ensures the consistency of these restrictions so that $\mathbb{P}_\rho$ defines a probability measure on Ω. We will consider two random processes $(x_n)_n$ and $(\rho_n)_n$ defined on $\omega = (i_0, i_1, \dots) \in \Omega$ by

$$x_n(\omega) = i_n,$$
$$\rho_n(\omega) = \frac{T_{i_n,i_{n-1}} \circ \cdots \circ T_{i_1,i_0}(\rho(i_0))}{\mathrm{Tr}\big[T_{i_n,i_{n-1}} \circ \cdots \circ T_{i_1,i_0}(\rho(i_0))\big]}.$$

Note that the variable ρ_n is a state on $\mathfrak{h}_{x_n}$, and that the process $(x_n, \rho_n)_n$ is Markov, corresponding to the transitions defined loosely as follows: conditionally on $(x_n = j, \rho_n = \eta)$, one has

$$(x_{n+1}, \rho_{n+1}) = \left(i, \frac{T_{i,j}(\eta)}{\mathrm{Tr}\,[T_{i,j}(\eta)]}\right) \quad \text{with probability } \mathrm{Tr}\,[T_{i,j}(\eta)].$$

Remark that $(x_n)_n$ or $(\rho_n)_n$ are not Markov processes. For notational simplicity, we will often consider an initial state ρ of the form $\rho \otimes |i\rangle\langle i|$. The corresponding probability $\mathbb{P}_\rho$ will then be denoted by $\mathbb{P}_{i,\rho}$.

As noted in [APSS12], classical Markov chains can be written as open quantum walks. Consider for example a Markov chain $(M_n)_n$ on the vertex set V, with probability $t_{i,j}$ of transition from j to i and initial distribution $(p_i)_{i\in V}$. Define the open quantum walk $\mathfrak{M}$ with $\mathfrak{h}_i \equiv \mathbb{C}$ and $T_{i,j} = \sqrt{t_{i,j}} I_{\mathbb{C}}$. If the initial state is $\rho = \sum_{i\in V} p_i \otimes |i\rangle\langle i|$, then $\mathfrak{M}(\rho)$ is of the form

$$\mathfrak{M}(\rho) = \sum_{i\in V} \big(\sum_{j\in V} t_{i,j} p_j\big) \otimes |i\rangle\langle i|.$$

Therefore, x_0 has the same law as M_0 and x_1 has the same law as M_1, etc. This OQW will be called the *minimal dilation of the Markov chain* (because it is an OQW implementation of the Markov chain with minimal spaces $\mathfrak{h}_i$).

II Open Quantum Walks and non-commutative Dirichlet forms

In this section we introduce the non-commutative Dirichlet forms associated to the OQW. Dirichlet forms have been extensively studied in classical probability theory since their introduction by A; Beurling and J. Deny in two seminal papers [BD] [BD59]. In the non-commutative setting, they were first considered by Davies and Lindsay in [DLa] and then studied by several authors [DLb] [Cip08] [Cip98]. Our motivation is to solve the Dirichlet problem defined in the next sections as the observables that minimize a certain functional. This will be done in the next section.

In this section we focus on the definition of the Dirichlet form and we study of its properties. We prove the following: there exists a one-to-one correspondence between a certain class of non-commutative Dirichlet forms and OQW. This provides an analogue of a result of Beurling and Deny.

We assume all over this section that $\pi = \sum_{i\in V}\pi(i)\otimes|i\rangle\langle i|$ is an invariant state for the OQW $\mathfrak{M}$, which furthermore is *faithful*: $\pi > 0$. Remark that $(\mathrm{Tr}[\pi(i)])_{i\in V}$ is an invariant measure of the process $(x_n)_{n\geq 0}$, in the sense that the law of x_n conditioned on $\rho_0 = \pi$ is $(\mathrm{Tr}[\pi(i)])_{i\in V}$.

II. 1 Characterization of OQW in terms of non-commutative Dirichlet forms

We now focus on the definition of the Dirichlet form and its properties. Define on $\mathcal{B}(\mathcal{H})$ the scalar product

$$\langle X, Y\rangle_\pi = \mathrm{Tr}\left[\pi^{1/2}X^*\pi^{1/2}Y\right], \qquad X, Y \in \mathcal{B}(\mathcal{H}). \tag{II.1}$$

For latter discussions we also need the Hilbert-Schmidt scalar product defined by:

$$\langle X, Y\rangle_{HS} = \mathrm{Tr}\left[X^*Y\right], \qquad X, Y \in \mathcal{B}(\mathcal{H}).$$

Definition II.1. *The* Dirichlet form *associated to the OQW $\mathfrak{M}$ is the quadratic form:*

$$\mathcal{E}(X,Y) = \langle X, (I-\mathfrak{M}^*)(Y)\rangle_\pi = \mathrm{Tr}\left[\pi^{1/2}X^*\pi^{1/2}(I-\mathfrak{M}^*)(Y)\right], \quad X \in \mathcal{B}(\mathcal{H}). \tag{II.2}$$

The central hypothesis in the following is the detailed balance condition define below which ensures the positivity of the Dirichlet form

Definition II.2. *We say that the OQW $\mathfrak{M}$ satisfies the* Detailed Balance Condition *(DBC) with respect to π if $\mathcal{M}^*$ is selfadjoint with respect to the scalar product $\langle\cdot,\cdot\rangle_\pi$. In particular, it implies that for all $i,j \in V$:*

$$T_{i,j}(\cdot) = \pi(i)^{1/2} T^*_{i,j}\left(\pi(j)^{-1/2} \cdot \pi(j)^{-1/2}\right)\pi(i)^{1/2}. \tag{II.3}$$

The different properties that characterize the Dirichlet form are summarized in Theorem II.1. The main property that ensured that the map is Markovian is the contractivity of the form with respect to the projection on a certain cone. For this reason we need the following notation: the Hilbertian projection of a selfadjoint operator $X \in \mathcal{B}(\mathcal{H})$ into the convex $I - \mathcal{B}(\mathcal{H})_+$ and with respect to the scalar product $\langle\cdot,\cdot\rangle_\pi$ is written $X \wedge I$. We will need its explicit formula:

Lemma II.1. *$X \wedge I$ is given by the formula*

$$\begin{aligned} X \wedge I &= I - \pi^{-1/4}\left(\pi^{1/2} - \pi^{1/4} X \pi^{1/4}\right)_+ \pi^{-1/4} \\ &= I - (I - X)_+ \end{aligned} \tag{II.4}$$

where X_+ is the positive part of X.

Proof. The projection $X \wedge I$ on the convex cone $I - \mathcal{B}(\mathcal{H})_+$ is the unique element of $I - \mathcal{B}(\mathcal{H})_+$ such that the following inequality holds:

$$\langle X - X \wedge I, Y - X \wedge I\rangle_\pi \le 0 \qquad \forall Y \in I - \mathcal{B}(\mathcal{H})_+. \tag{II.5}$$

Let compute the left-hand side. For all $Y \in \mathcal{B}(\mathcal{H})_+$, we have:

$$\begin{aligned} &\langle X - X \wedge I, (I - Y) - X \wedge I\rangle_\pi \\ &\quad = \mathrm{Tr}\left\{\left[\pi^{1/4} X \pi^{1/4} - \pi^{1/4}(X \wedge I)\pi^{1/4}\right]\left[\pi^{1/2} - \pi^{1/4} Y \pi^{1/4} - \pi^{1/4}(X \wedge I)\pi^{1/4}\right]\right\} \\ &\quad = \mathrm{Tr}\left\{\left[\left(\pi^{1/2} - \pi^{1/4} X \pi^{1/4}\right) - \left(\pi^{1/2} - \pi^{1/4}(X \wedge I)\pi^{1/4}\right)\right] \times \right. \\ &\qquad\qquad \left. \times \left[\pi^{1/4} Y \pi^{1/4} - \left(\pi^{1/2} + \pi^{1/4}(X \wedge I)\pi^{1/4}\right)\right]\right\} \end{aligned}$$

As the orthogonal projection of $\pi^{1/2} - \pi^{1/4} X \pi^{1/4}$ on $\mathcal{B}(\mathcal{H})_+$ for the Hilbert-Schmidt scalar product is its positive part and $\pi^{1/4} Y \pi^{1/4} \in \mathcal{B}(\mathcal{H})_+$, we get that:

$$\pi^{1/2} - \pi^{1/4}(X \wedge I)\pi^{1/4} = \left(\pi^{1/2} - \pi^{1/4} X \pi^{1/4}\right)_+,$$

which is the first equality. The second equality is straightforward, once we remark that $(\pi^{1/4}X\pi^{1/4})_+ = \pi^{1/4}X_+\pi^{1/4}$ for all $X \in \mathcal{B}(\mathcal{H})$. To see this write $\pi^{1/4}X\pi^{1/4} = \pi^{1/4}X_+\pi^{1/4} - \pi^{1/4}X_-\pi^{1/4}$ and use the unicity of the decomposition between a positive and a negative part. □

By the same computation we can show that $I-(I-X)_+$ is the orthogonal projection of X on $I-\mathcal{B}(\mathcal{H})_+$, but for the Hilbert-Schmidt scalar product. It is remarkable that both projections coincide.

We now give our main result on the characterization of OQW in terms of non-commutative Dirichlet forms.

Theorem II.1. *There is a one to one correspondence between generalized OQW $\mathfrak{M}$ which satisfies the DBC with respect to π and symmetric real sesquilinear form $\mathcal{E}$ on $\mathcal{B}(\mathcal{H})$ that satisfies properties 1. to 5. listed below. This correspondence is given by*

$$\mathcal{E}(X,Y) = \langle X, (I-\mathfrak{M}^*)(Y)\rangle_\pi. \tag{II.6}$$

The different properties of the Dirichlet form are:

1. *$\mathcal{E}$ is positive, in the sense that for all $X \in \mathcal{B}(\mathcal{H})$:*

$$\mathcal{E}(X) := \mathcal{E}(X,X) \geq 0 \tag{II.7}$$

for all selfadjoint operator $X \in \mathcal{B}(\mathcal{H})$.

2. *$\mathcal{E}$ satisfies the* Dirichlet condition*: for all selfadjoint operator $X \in \mathcal{B}(\mathcal{H})$,*

$$\mathcal{E}(X \wedge I) \leq \mathcal{E}(X). \tag{II.8}$$

3. *$\mathcal{E}_n$ satisfy the above Dirichlet condition (II.8) for all $n \geq 0$, where $\mathcal{E}_n$ is the real sesquilinear form on $\mathcal{B}(\mathcal{H}) \otimes \mathfrak{M}_n(\mathbb{C})$ defined by:*

$$\mathcal{E}_n\left[(X_{ij})_{1\leq i,j\leq n}\right] := \sum_{i,j=1}^{n} \mathcal{E}(X_{ij}), \qquad (X_{ij})_{1\leq i,j\leq n} \in \mathcal{B}(\mathcal{H}) \otimes \mathfrak{M}_n(\mathbb{C}). \tag{II.9}$$

4. *$\mathcal{E}$ is* conservative*: $\mathcal{E}(I) = 0$.*

5. *This last property is specific to OQW quantum channels:*

$$\mathcal{E}(X) = \mathcal{E}(\mathcal{A}(X)) + \|(I-\mathcal{A})(X)\|_\pi^2 \qquad \text{for all } X \in \mathcal{B}(\mathcal{H}). \tag{II.10}$$

Before going to the proof, we would like to emphasize that this theorem is not specific to OQW quantum channels.Indeed, for any quantum channels on $\mathcal{B}(\mathcal{H})$ with a faithful invariant state it is possible to define a Dirichlet form by Equation (II.2) and the previous theorem still holds, without Property 5. which ensures that $\mathfrak{M}$ is an OQW, which comes from Lemma I.1.

Proof. We first prove that, given an OQW $\mathfrak{M}$, the Dirichlet form satisfies properties 1. to 5.

By the DBC condition, $I - \mathfrak{M}^*$ is a positive operator on the Hilbert space $\mathcal{B}(\mathcal{H}), \langle\cdot,\cdot\rangle_\pi$, which implies property 1.

For property 2, remark that $\mathcal{B}(\mathcal{H})_+$ is stable by $\mathcal{M}^*$ and so is $I - \mathcal{B}(\mathcal{H})_+$. Thus, for all selfadjoint operator $X \in \mathcal{B}(\mathcal{H})$, $\mathfrak{M}^*(X \wedge I) \in I - \mathcal{B}(\mathcal{H})_+$ and using Inequality (II.5) we get:

$$\mathcal{E}(X - X \wedge I, X \wedge I) = -\langle X - X \wedge I, \mathfrak{M}^*(X \wedge I) - X \wedge I\rangle \geq 0.$$

Then:

$$\mathcal{E}(X - X \wedge I, X + X \wedge I) = \mathcal{E}(X - X \wedge I, X - X \wedge I) + 2\mathcal{E}(X - X \wedge I, X \wedge I) \geq 0.$$

The left-hand side is equal to $\mathcal{E}(X) - \mathcal{E}(X \wedge I)$ so this is Inequality (II.8). To prove 3, we apply 2. to the OQW $\mathcal{M}_n := \mathcal{M} \otimes I_n$ acting on $\mathcal{H} \otimes \mathbb{C}^n$, where the set of vertices is still V but the internal Hilbert spaces are now $\mathfrak{h}_i \otimes \mathbb{C}^d$. The Dirichlet form associated to $\mathcal{M}_n$ is exactly the one given by Equation (II.9).

Finally property 4. is obvious and property 5. comes from Lemma I.1.

We now prove the converse. Suppose that $\mathcal{E}$ is a sesquilinear form on $\mathcal{B}(\mathcal{H})$ that satisfies properties 1. to 5. and define $\mathfrak{M}$ by Equation (II.6). Because of property 5. and Lemma I.1, we only need to prove that $\mathfrak{M}$ is a trace-preserving quantum channel, that is, it is completely positive and $\mathcal{M}^*(I) = I$.

Let prove that $\mathfrak{M}^*$ is positive. The fact that it is completely positive will follow by the same argument using property 3.Let define for all $\alpha > 0$ the forms $\mathcal{E}_\alpha$ defined by:

$$\mathcal{E}_\alpha(X, Y) = \mathcal{E}(X, Y) + \alpha\langle X, Y\rangle_\pi, \qquad X, Y \in \mathcal{B}(\mathcal{H}).$$

Remark that the $\mathcal{E}_\alpha$ are all real sesquilinear forms on $\mathcal{B}(\mathcal{H})$. Moreover, for all $\alpha > 0$, $\mathcal{E}_\alpha$ is a *coercive form*, i.e. there exists $K > 0$ such that for all selfadjoint operators $X \in \mathcal{B}(\mathcal{H})$:

$$\mathcal{E}_\alpha(X) \geq K \|X\|_\pi^2 . \tag{II.11}$$

Indeed, for all $\alpha > 0$ and all $X \in \mathcal{B}(\mathcal{H})$,

$$\mathcal{E}_\alpha(X) = \mathcal{E}(X) + \alpha \|X\|_\pi^2 .$$

Consequently, by an usual argument (see for instance the proof of Lax-Milgram Theorem [Bré83]), $(\alpha+1)I+\mathfrak{M}^*$ is invertible.

Take $Y \in \mathcal{B}(\mathcal{H})$ such that $0 \le Y \le I$. In particular, $I-Y$ is positive, so that by Lemma II.1, Y is in the convex $I-\mathcal{B}(\mathcal{H})_+$ and $Y = Y \wedge I$. Write $X = \alpha((1+\alpha)I-\mathfrak{M}^*)^{-1}(Y)$. We have:

$$\begin{aligned}
&\|X - X\wedge I\|_\pi^2 + \langle X - X\wedge I, X\wedge I - Y\rangle_\pi \\
&\quad= \langle X - X\wedge I, X - Y\rangle_\pi \\
&\quad= -\frac{1}{\alpha}\mathcal{E}(X - X\wedge I, X) \\
&\quad= -\frac{1}{2\alpha}\big(\mathcal{E}(X - X\wedge I, X + X\wedge I) + \mathcal{E}(X - X\wedge I, X - X\wedge I)\big)
\end{aligned}$$

By property 1. the last term in this last equality is positive. The first term is equal to $\mathcal{E}(X) - \mathcal{E}(X\wedge I)$ which is also positive by property 3. Furthermore, as $Y \in I - \mathcal{B}(\mathcal{H})_+$, $\langle X - X\wedge I, X\wedge I - Y\rangle_\pi \ge 0$ so that $X = X\wedge I$. We just proved that $R_\alpha := ((1+\alpha)I - \mathfrak{M}^*)^{-1}$ is positive for all $\alpha > 0$. The result follows using the formula:

$$\mathfrak{M}^* = \lim_{\alpha\to+\infty}(1+\alpha)(R_\alpha - I).$$

Finally, applying this to $Y = I$ with property 4, we get that $\mathfrak{M}^*(I) = I$.

We finish the proof by checking that π is an invariant state. Indeed, using the fact that $\mathfrak{M}^*$ is selfadjoint for $\langle\cdot,\cdot\rangle_\pi$ and Equation (II.3), we get that

$$\pi = \pi^{1/2}\mathfrak{M}^*(I)\pi^{1/2} = \mathfrak{M}(\pi).$$

□

II. 2 Link with Dirichlet forms on standard representation of von Neumann algebras:

The previous paragraph is in fact a simple case of the more general theory of non-commutative Dirichlet forms developed for instance by Cipriani [Cip08] for the symmetric case and Guido, Isola, Scarlatti [GIS96] for the non-symmetric case. In this paragraph we show how our results stick with the usual theory.

The general theory of non-commutative Dirichlet form is written in terms of a quadratic functional over the Hilbert space of the standard representation of a von Neumann algebra. In our case, the von Neumann algebra is the whole algebra equipped with the invariant state: $(\mathcal{B}(\mathcal{H}),\pi)$. Its standard representation is given by the Hilbert

space $\mathcal{K} = \mathcal{B}(\mathcal{H})$ with modular cone $\mathcal{K}_+ = \mathcal{B}(\mathcal{H})_+$, provided with the Hilbert-Schmidt scalar product $\langle\cdot,\cdot\rangle_{HS}$. The conjugation is the usual adjoint: $X \mapsto X^*$.
The crucial point is the canonical injection of $\mathcal{B}(\mathcal{H})$ into $\mathcal{K}$ (that is into itself) that preserves the positive cone induced by π. It is given by:

$$i : X \mapsto \pi^{1/4} X \pi^{1/4}. \tag{II.12}$$

This injection can also be seen as the natural injection of the L^∞ space $\mathcal{B}(\mathcal{H})$ into the L^2 space $(\mathcal{B}(\mathcal{H}), \langle\cdot,\cdot\rangle_\pi)$. Now we see that the Dirichlet form as defined in Definition II.1 is also given by:

$$\mathcal{E}(X,Y) = \langle i(X), (I - \mathfrak{M}^*)(i(Y))\rangle_{HS} := \mathcal{E}_{\mathcal{K}}(i(X), i(Y)), \qquad X, Y \in \mathcal{B}(\mathcal{H}).$$

Finally, in terms of the Hilbert-Schmidt scalar product, $X \wedge I$ is actually the orthogonal projection of $i(X)$ on the convex cone $\pi^{1/2} - \mathcal{K}_+$.

III Variational approach to the Dirichlet problem

In this section we will study the existence and uniqueness of solutions to the Dirichlet problem

$$(\mathrm{Id} - \mathfrak{M}^*)(Z) = A,$$

either on the full graph V, or on a finite subdomain $D \subset V$. The meaning of this will be made precise below. This is only possible under the DBC condition (definition II.2). We start by defining precisely the boundary of D and by introducing some notations.

For an OQW $\mathfrak{M}$ and $i \in V$ we define the *neighborhood* $n(i)$ of i to be the set

$$n(i) = \{j \in V \,|\, L_{j,i} \neq 0\},$$

and for any subset D of V we define its boundary ∂D and interior $\operatorname{int} D$ as the sets

$$\partial D = \{i \in V \,|\, n(i) \not\subset D\}$$
$$\operatorname{int} D = D \setminus \partial D.$$

III. 1 Dirichlet problem on the whole domain

We start with the case of the Dirichlet problem on V.

Definition III.1. *An operator* $A = \sum_{i\in V} A_i \otimes |i\rangle\langle i|$, *with* $x_i \in \mathcal{B}(\mathfrak{h}_i)$ *for each* $i \in V$, *is called* Quantum Harmonic *if it satisfies* $\mathfrak{M}^*(A) = A$, *that is, if for any* $j \in V$ *we have*

$$A_j = \sum_{i\in V} T_{i,j}(A_i).$$

In the case of a minimal dilation of a classical Markov chain, this definition is equivalent to the classical definition of harmonicity.

Quantum harmonic observables are easily characterized in term of the Dirichlet form.

Proposition III.1. *Suppose that* $\mathfrak{M}$ *has the DBC. Then* X *is a quantum harmonic observable if and only if* $\mathcal{E}(X)$ *is minimal, if and only if* $\mathcal{E}(X) = 0$.

Proof. This is trivial, as the DBC implies that $\mathcal{E}(X) \geq 0$ with equality if and only if $(I - \mathfrak{M}^*)(X) = 0$, that is, if X is harmonic. □

III. 2 Dirichlet problem on a sub-domain

We now focus on the Dirichlet problem on a finite domain $D \subset V$. We define respectively the inner data A and the outer data B as

$$A = \sum_{j\in \mathrm{int}\, D} A_j \otimes |j\rangle\langle j| \qquad B = \sum_{j\in\partial D} B_j \otimes |j\rangle\langle j|,$$

where $A_j, B_j \in \mathfrak{h}_j$ for all j.

The main hypothesis concerns the spectrum of $\mathcal{M}$.

Definition III.1. *We say that* $\mathcal{M}$ *has a* Spectral Gap *if there exists a constant* $\lambda > 0$ *such that the following inequality (Poincaré Inequality) holds for all selfadjoint operator* $X \in \mathcal{B}(\mathcal{H})$

$$\lambda\, Var_\pi(X) \leq \mathcal{E}(X), \tag{III.1}$$

where Var_π *is the variance defined by*

$$Var_\pi(X) = \|X - Tr[\pi X] I\|_\pi^2, \qquad X \in \mathcal{B}(\mathcal{H}).$$

In general, we write $\lambda(\mathfrak{M})$ *the best constant* $\lambda \leq 0$ *such that Inequality* (III.1) *holds, so that* $\mathfrak{M}$ *has a spectral gap if and only if* $\lambda(\mathfrak{M}) > 0$.

One of the consequence of this definition is that if $\lambda(\mathfrak{M}) > 0$, then $\mathfrak{M}$ is *irreducible*, in the sense that Ker $\mathfrak{M}^* = \mathbb{C}I$.

Our main theorem is the following.

Theorem III.1. *Suppose that $\mathfrak{M}$ has the DBC and that $\lambda(\mathfrak{M}) > 0$. Assume additionally that $Tr[\pi A] = 0$. Then there is a unique solution to the* Dirichlet problem

$$\begin{cases} (\mathrm{Id} - \mathfrak{M}^*)(Z)(k) & = A(k) \quad \textit{for } k \in D, \\ Z(k) & = B(k) \quad \textit{for } k \in \partial D. \end{cases} \tag{III.2}$$

Furthermore, this solution is the unique minimizer over the set $B + \bigoplus_{j \in \mathrm{int}\, D} \mathcal{B}(\mathfrak{h}_j)$ of the following functional

$$E_1(X) := \frac{1}{2}\mathcal{E}(X) - \langle A, X\rangle_\pi. \tag{III.3}$$

Proof. We write $K = B + \bigoplus_{j \in \mathrm{int}\, D} \mathcal{B}(\mathfrak{h}_j)$. It is a non-empty closed convex set. First remark that Z is a solution of the Dirichlet problem iff $Z \in K$ and for all $Y \in \bigoplus_{j \in \mathrm{int}\, D} \mathcal{B}(\mathfrak{h}_j)$,

$$\mathcal{E}(Z, Y) = \langle A, Y\rangle_\pi$$

(it means that Z is a weak solution to the Dirichlet problem). In particular, $Z \in K$ is a solution iff for all $Y \in K$,

$$\mathcal{E}(Z, Y - Z) \geq \langle A, Y - Z\rangle_\pi.$$

Now taking $\tilde{Y} = Z \pm Y$ in the above inequality we see that it is actually a sufficient condition to find back the previous equality. Consequently this inequality is a necessary and sufficient condition on Z to be a solution.

At this point, we would like to apply Stampacchia Theorem [Bré83] to conclude the proof. However $\mathcal{E}$ is not coercive on the Hilbert space $(\mathcal{B}(\mathcal{H}), \langle\cdot,\cdot\rangle_\pi)$. Consequently we need to restrict the problem to a smaller space.

Write $L = I - \mathfrak{M}^*$. As $\mathfrak{M}$ is irreducible, $\mathcal{K} = \mathrm{Im}\ L = (\mathbb{C}I)^\perp$ is stable by L and the restriction of this latter to $\mathcal{K}$ is invertible. We write M this restriction. As $\mathfrak{M}$ has the DBC, M is selfadjoint for $\langle\cdot,\cdot\rangle_\pi$. On $\mathcal{K}$, we define the real-sesquilinear form

$$\widetilde{\mathcal{E}}(X, Y) = \langle X, MY\rangle_\pi, \qquad X, Y \in \mathcal{K}.$$

We claim that $\widetilde{\mathcal{E}}$ is coercive. Indeed, for any $X \in \mathcal{K}$, $\mathrm{Tr}[\pi X] = 0$ as X is orthogonal to the identity operator. Consequently the Poincaré Inequality (III.1) is exactly the coercivity condition.

Consequently, repeating the above argument with M on $\mathcal{K}$, there exists a unique solution Z_1 to the equation:

$$\begin{cases} (\mathrm{Id} - \mathfrak{M}^*)(Z_1)(k) & = A(k) \quad \text{for } k \in D, \\ Z_1(k) & = B(k) - \mathrm{Tr}[\pi B] I_{\mathfrak{h}_k} \quad \text{for } k \in \partial D. \end{cases} \tag{III.4}$$

This solution belongs to $\mathcal{K} \cap K$ and is the unique minimizer on this set of :

$$\widetilde{E}_1(X) := \frac{1}{2}\widetilde{\mathcal{E}}(X) - \langle A, X\rangle_\pi.$$

Now remark that $\widetilde{E}_1$ and E_1 actually coincide on $\mathcal{K}$ as $A \in (\mathbb{C}I)^\perp$. Thus, defining

$$Z = Z_1 + \mathrm{Tr}[\pi B] I_{\mathcal{H}},$$

one check that it is the unique solution of Equation (III.2) and for all $X \in K$,

$$E_1(Z) = \widetilde{E}_1(Z_1) \leq \widetilde{E}_1\left(X - \mathrm{Tr}[\pi X] I_{\mathcal{H}}\right) = E_1(X),$$

Furthermore, if $\widetilde{Z}$ is a minimizer of E_1 on K, then $\widetilde{Z} - \mathrm{Tr}[\pi\widetilde{Z}]I_{\mathcal{H}} = Z - \mathrm{Tr}[\pi Z]I_{\mathcal{H}}$. As $Z - \widetilde{Z} \in \bigoplus_{j\in \mathrm{int}\, D} \mathcal{B}(\mathfrak{h}_j)$, we get $Z = \widetilde{Z}$ wich concludes the proof. □

Part C

Environment Induced Decoherence

The next chapter presents two results concerning Environment-Induced Decoherence (EID) for Quantum Dynamical Semigroup.

The first one states that EID always occurs on finite von Neumann algebra, when the quantum Markov semigroup has a faithful normal invariant state. This part is a collaboration with Yan Pautrat. The main point in this result is the simplicity of the proof. Whereas previous known results on EID invoke strong analysis of the the semigroup and its different extensions, here we use powerful results from operator spaces theory which allows to reduce the problem to simple computations. In the same spirit, I shall give a simple recipe in order to study the structure of the normal invariant states of a quantum Markov semigroup, which, when applied to the finite-dimensional case, gives back already known results.

The second result of this chapter involves the definition of adapted forms of variance and relative entropy, in order to study the speed of the decoherence, by use of functional inequalities. I have greatly benefited from long discussions with Raffaella Carbone concerning this part.

Both parts are still works-in-progress and consequently have not been yet submitted for publication. In the conclusion I detail the points I think would need to be developed before doing so.

Chapter 5

Environment Induced Decoherence for quantum Markov semigroups

Introduction

Until now in this thesis, we focused on which classical processes can appear in quantum dynamics, avoiding the question of how they appear. Nonetheless this is a central question in quantum mechanics, ever since the premise of the theory and the Schrödinger's cat dilemma. This one can be formulated as follows:

> Imagine a box with a cat inside. In the box, the experimenter puts a very small portion of radioactive elements, just enough so that the probability for an atom to decay in one hour is one half. Additionally he places a Geiger counter in the box with a mechanism such that, if the counter clicks, the cat is brutally killed.

If the whole box is subject to the quantum laws, then the state of the cat, either alive or dead, must be correlated with the state of the radioactive elements. Subsequently, after one hour, it should be a superposition of the two states: alive and dead. Of course this is not satisfactory at all because the cat cannot be simultaneously dead and alive, independently of an outside observer. The problem that emerges is the one of the frontier between classical and quantum world. One can for instance wonder at which scales the laws of physics stop being quantum and become classical.

The physical explanation which has emerged in the past thirty years is known as Environment-Induced Decoherence. It consists in getting around the spatial problem of finding the frontier by a dynamical explanation. In the pioneering articles [Zur81] [Zur82], Zurek proposed the following interpretation of the Schrödinger's cat dilemma. A quantum system is never perfectly isolated and quantum correlations, that is, the existence of Schrödinger's cat-like states, disappear due to the action of the environment on the system. The obvious fact that in our everyday experience we do not see quantum behavior in the real world is just the consequence of how fast this decoherence process takes place.

In this chapter we focus on open quantum systems modeled by quantum Markov semigroups. In view of the above discussion, it is an important problem to define in this framework the appropriate notion of Environment-Induced Decoherence. The first mathematical formulation in this context is due to Blanchard and Olkiewicz [BO03].

We consider a quantum Markov semigroup $(\mathcal{P}_t)_{t\geq 0}$ on some von Neumann algebra $\mathcal{M}$, acting on some Hilbert space $\mathcal{H}$. That is, $\mathcal{P}$ is a weak* continuous semigroup of completely positive maps. Blanchard and Olkiewicz proposed the following formulation of Environment-Induced Decoherence, which in this form is originally from [Hel11].

Definition 1. *We say that there is* Environment-Induced Decoherence *(EID) if there exists a decomposition of $\mathcal{M}$ as*

$$\mathcal{M} = \mathcal{M}_0 \oplus \mathcal{M}_1 \tag{.1}$$

such that:

- *both $\mathcal{M}_0$ and $\mathcal{M}_1$ are $\mathcal{P}$-invariant.*
- *$\mathcal{M}_0$ is the maximal algebra on which $\mathcal{P}$ acts as a $*$-automorphism.*
- *$\mathcal{M}_1$ is a non-empty, $*$-weakly closed and $*$-invariant subspace of $\mathcal{B}(\mathcal{H})$.*
- *$\mathcal{P}_t(X)$ $*$-weakly converges to 0 if $X \in \mathcal{M}_1$.*

We call $\mathcal{M}_0$ the algebra of effective observables*.*

This definition has the following interpretation. The space $\mathcal{M}_1$ is thought as the part of the system which is beyond experimental resolution. Indeed, if the decoherence is fast enough, any measurement of an observable $X \in \mathcal{M}_1$ will give the value 0. Thus in the long-time asymptotic, the system behaves effectively like a closed system described by the von Neumann algebra $\mathcal{M}_0$.

A simple and significant example of such a decomposition is to take $\mathcal{H}$ finite-dimensional, $\mathcal{M}_0$ the algebra of operators diagonal in a fixed orthonormal basis and $\mathcal{M}_1$ the set of off-diagonal operators, that is with zero-entries on the diagonal. Then if such a decomposition holds for a QMS $\mathcal{P}$, asymptotically the observables all become diagonal so that quantum correlations disappear.

Environment-Induced Decoherence was already proven under different hypotheses, the most general one being:

- $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and the semigroup is uniformly continuous with a faithful normal invariant state [DFSU14];
- $\mathcal{M}$ is any von Neumann algebra and the semigroup has a faithful (sub-)invariant weight *whose modular group commutes with the semigroup* [ŁO03].

Hence, in the case of a von Neumann algebra which is not of type I, the most general result assume the existence of a (sub)invariant weight whose modular group commutes with the semigroup. This last assumption is purely technical and it is not difficult to find quantum Markov semigroups with faithful invariant state which does not have this property[1].

In this chapter we prove the following. Under the assumption that $\mathcal{M}$ is a *finite* von Neumann algebra (or equivalently it has a normal tracial state) and that $\mathcal{P}$ has a faithful normal invariant state, then EID occurs. Thus we do not need to assume that the modular group associated to the invariant state commute with the semigroup. We also state more general assumptions under which the same proof still holds.

Our proof relies on the existence of a certain invariant state, which is tracial on the algebra of effective observables and on a result of Takesaki on normal conditional expectation.

Mathematically, Environment-Induced Decoherence goes beyond the theoretical interpretation presents above, as it describes the long-time asymptotic behavior of quantum open systems. In this sense it is very close to the Return to Equilibrium property. This latter property is closely related to the structure of the invariant states of the quantum Markov semigroup. Indeed one of the main feature of return to equilibrium is the existence of a projection on the set of fixed points. Whenever the semigroup has a faithful normal invariant state, this set is in fact an algebra, whose properties are directly reflected on the invariant states via the projection (in this case a normal conditional expectation). In fact, the existence of this conditional expectation plays a fundamental

[1] A well-known example of a class of QMS with this property is the generic quantum Markov semigroups [AK02].

role in our proof of Environment-Induced Decoherence.

The structure of the invariant states in itself is the subject of a lot of interest [DFSU14] [BN12] [CP15]. In the case where $\mathcal{M} = \mathcal{B}(\mathcal{H})$ the structure is already known, so that the question now is to find tools to study more general situations. In this chapter, I propose a recipe which generalized the known results, in the sense that when applied to the finite-dimensional case we recover the usual result. My goal in exposing this recipe is to show that the powerful tools of operator space theory can be successfully used in the study of Markovian quantum systems.

We saw that the speed at which the decoherence process occurs is a central matter. First at the theoretical level, in order to better understand the frontier between the classical and the quantum world, but also at the practical level, because if decoherence is seen as an obstruction to the engineering of quantum computers, the time of the process gives the potential window of time for performing quantum operations.

In this chapter I make a first step toward the study of the speed of decoherence for quantum Markov semigroups. I propose, for finite dimensional systems, adapted definitions of variance and relative entropy. They are meant to measure the distance to the algebra of effective observables. It is then possible to define analogues of the Poincaré and log-Sobolev Inequalities and to prove exponential rate of the decoherence.

This chapter is structured as follows. In section I we recall some notions about Return to Equilibrium and propose a recipe to study the structure of the normal invariant states of a quantum Markov semigroup. In section II we prove our main result on decoherence on finite von Neumann algebra. In section III we study the speed of the decoherence using functional inequalities.

I Return to Equilibrium and the structure of the invariant states

In this section we recall basic results on Return to Equilibrium and the structure of the invariant states. In subsection I. 1 we define return to equilibrium and we state the main sufficient condition that ensures this asymptotic property. In subsection I. 2 we explain a general recipe to study the structure of normal invariant states. We apply this recipe in subsection I. 3 to the case of a finite dimensional Hilbert space.

I. 1 Return to Equilibrium of Open Quantum System

First we recall the definition of Return to Equilibrium.

Definition I.1. *We say that the system has the* Return to Equilibrium *property (RE) if for all normal functional $\varphi \in \mathcal{M}_*$ and all $X \in \mathcal{M}$:*

$$\lim_{t\to 0} \varphi\left(\mathcal{P}_t(X)\right) = \varphi(I)\rho(X), \tag{I.1}$$

for a certain $\mathcal{P}$-invariant normal state ρ (that depends on φ).

The central object in the study of (RE) is the set of fixed points $\mathcal{F}(\mathcal{P})$ defined as

$$\mathcal{F}(\mathcal{P}) = \{X \in \mathcal{M},\ \mathcal{P}_t(X) = X \text{ for all } t \geq 0\}. \tag{I.2}$$

Under the assumption that there exists a faithful normal invariant state σ, we know from Evans [Eva77] that $\mathcal{F}(\mathcal{P})$ is a subalgebra of $\mathcal{M}$. Moreover, from Frigerio [Fri78], there exists a unique normal conditional expectation $E_{\mathcal{F}}$ from $\mathcal{M}$ to $\mathcal{F}(\mathcal{P})$ compatible with σ, that is, such that $\sigma = \sigma \circ E_{\mathcal{F}}$. Later we need an explicit formula for this conditional expectation:

$$E_{\mathcal{F}}(X) = \lim_{\lambda\to 0} \lambda \int_0^{+\infty} e^{-\lambda t}\mathcal{P}_t(X)dt. \tag{I.3}$$

Then (RE) can be rephrased in terms of the algebra $\mathcal{F}(\mathcal{P})$ and the conditional expectation $E_{\mathcal{F}}$:

Lemma I.1. *Return to Equilibrium is equivalent to the existence of the following limit:*

$$w^* - \lim_{t\to +\infty} \mathcal{P}_t(X) = E_{\mathcal{F}}(X) \textit{ for all } X \in \mathcal{M}. \tag{I.4}$$

Proof. Assume that RE holds and take a normal functional $\varphi \in \mathcal{M}_*$. We assume without loss of generality that $\varphi(I) = 1$. Then, using the form (I.3) of the conditional expectation and Equation (I.1), we see that $\rho = \varphi \circ E_{\mathcal{F}}$. As it holds for all normal functional, we obtain the limit (I.4).

Conversely, assume that the limit (I.4) holds and take a normal functional $\varphi \in \mathcal{M}_*$. Then we see that the limit (I.1) is true with $\rho = \varphi \circ E_{\mathcal{F}}$. □

We shall see later that the Environment-Induced Decoherence property is similar to Equation (I.4), in the sense that it can be expressed in terms of the existence of a compatible conditional expectation on the algebra of effective observables $\mathcal{M}_0$. An other algebra, called the *Decoherence-Free algebra* (for a reason that will become clear in the next section), is actually intimately related to RE. It is defined as:

$$\begin{aligned}\mathcal{N}(\mathcal{P}) = \Big\{X \in \mathcal{M},\ \mathcal{P}_t(X^*Y) = \mathcal{P}_t(X)^*\mathcal{P}_t(Y),\ \mathcal{P}_t(Y^*X) = \mathcal{P}_t(Y)^*\mathcal{P}_t(X) \\ \text{for all } t \geq 0 \text{ and all } Y \in \mathcal{M}\Big\}.\end{aligned} \tag{I.5}$$

The most direct condition for proving RE is given in Theorem I.1. It was latter refined for semigroups in Lindblad form by Fagnola and Rebolledo in [FR08].

Theorem I.1 (Frigerio and Verri [FV82], Frigerio [Fri78])**.** *Suppose that $\mathcal{P}$ has a faithful normal invariant state σ. Then the inclusion $\mathcal{F}(\mathcal{P}) \subset \mathcal{N}(\mathcal{P})$ holds. Furthermore, equality implies the Return to Equilibrium property. If furthermore $\mathcal{M}$ is finite-dimensional, then (RE) is equivalent to $\mathcal{F}(\mathcal{P}) = \mathcal{N}(\mathcal{P})$.*

I. 2 Structure of the invariant states

Our goal in this subsection is to describe a very general recipe in order to study the structure of the normal invariant states of a given QMS $\mathcal{P}$ on some von Neumann algebra $\mathcal{M}$. In the next subsection we apply this recipe to the case where $\mathcal{H}$ is finite-dimensional. I believe however that the same technique could be used in more general situations. Unfortunately for the moment I do not have more sophisticated examples. In particular it would be appreciable to have an illustration on type II von Neumann algebras, for instance for QMS as defined in [MOZ98].

The starting point comes (again) from Frigerio [Fri77], who noticed that a normal state φ is $\mathcal{P}$-invariant if and only if

$$\varphi \circ E_{\mathcal{F}} = \varphi. \tag{I.6}$$

It means in particular that the structure of an invariant state is intimately related to the structure of the algebra $\mathcal{F}(\mathcal{P})$. The recipe relies on a result of Takesaki on conditional expectation (Corollary 1 in [Tak72]):

Theorem I.2. *Let $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$. Let $\mathcal{N}^c$ denote the relative commutant $\mathcal{N}' \cap \mathcal{M}$ of $\mathcal{N}$ in $\mathcal{M}$.*

Suppose there exists a conditional expectation E from $\mathcal{M}$ to $\mathcal{N}$ compatible with σ. Then the map

$$\pi : \sum_{i=1}^{n} x_i \otimes y_i \in \mathcal{N} \otimes \mathcal{N}^c \mapsto \sum_{i=1}^{n} x_i y_i \in \mathcal{M} \tag{I.7}$$

is extended to an isomorphism from $\mathcal{N} \overline{\otimes} \mathcal{N}^c$ [2] *to the subalgebra $\mathcal{R}(\mathcal{N},\mathcal{N}^c)$ of $\mathcal{M}$ generated by $\mathcal{N}$ and $\mathcal{N}^c$. Furthermore, $\sigma \circ \pi^{-1}$ is decomposed into the product state $\sigma_1 \otimes \sigma_2$, where σ_1 is the restriction of σ to $\mathcal{N}$ and σ_2 the restriction of σ to $\mathcal{N}^c$. Equivalently, we have*

$$\sigma(xy) = \sigma(x)\sigma(y) \text{ for all } x \in \mathcal{N}, y \in \mathcal{N}^c. \tag{I.8}$$

What does this result tell us on $\mathcal{F}(\mathcal{P})$? First we obtain the following proposition.

Proposition I.1. *Assume that $\mathcal{P}$ has a faithful normal invariant state and that $\mathcal{F}(\mathcal{P}) \cap \mathcal{F}(\mathcal{P})^c = \mathbb{C}I$. Then there exists a unique faithful normal state σ on $\mathcal{F}(\mathcal{P})^c$ such that for all $Y \in \mathcal{F}(\mathcal{P})^c$,*

$$E_{\mathcal{F}}(Y) = \sigma(Y) I_{\mathcal{H}}. \tag{I.9}$$

Furthermore, for all invariant state φ and all $X \in \mathcal{F}(\mathcal{P})$ and $Y \in \mathcal{F}(\mathcal{P})^c$, we have $\varphi(XY) = \varphi(X)\sigma(Y)$.

Proof. For all $X \in \mathcal{F}(\mathcal{P})$ and $Y \in \mathcal{F}(\mathcal{P})^c$, we have $E_{\mathcal{F}}(XY) = XE_{\mathcal{F}}(Y) = E_{\mathcal{F}}(Y)X$, so that $E_{\mathcal{F}}(Y)$ belongs to $\mathcal{F}(\mathcal{P})^c$. Consequently, if $\mathcal{F}(\mathcal{P}) \cap \mathcal{F}(\mathcal{P})^c = \mathbb{C}I$, then there exists a functional $\sigma \in \mathcal{F}(\mathcal{P})^c_*$ such that for all $Y \in \mathcal{F}(\mathcal{P})^c$, we have $E_{\mathcal{F}}(Y) = \sigma(Y)I$. Obviously σ is a faithful normal state, as $E_{\mathcal{F}}$ is a faithful normal conditional expectation.
Then, using Theorem I.2, for all invariant state φ and all $X \in \mathcal{F}(\mathcal{P})$ and $Y \in \mathcal{F}(\mathcal{P})^c$, we get

$$\varphi(XY) = [\varphi \circ E_{\mathcal{F}}](XY) = [\varphi \circ E_{\mathcal{F}}](X)\,[\varphi \circ E_{\mathcal{F}}](Y) = \varphi(X)\sigma(Y).$$

□

We obtain that the restriction of any invariant state on $\mathcal{F}(P)^c$ is unique when $\mathcal{F}(\mathcal{P})$ is a factor. It should now be possible to deduce the form of the invariant states by a spatial decomposition with respect to the commutative von Neumann algebra $\mathcal{F}(\mathcal{P}) \cap \mathcal{F}(\mathcal{P})^c$ [BR87].

In the next subsection we apply this recipe to the case of a finite-dimensional Hilbert space, where the structure of the algebras $\mathcal{F}(\mathcal{P})$ and $\mathcal{F}(\mathcal{P})^c$ are known.

[2] Here the symbol $\overline{\otimes}$ means the algebraic tensor product.

I. 3 The case of a finite-dimensional Hilbert space

In this subsection we assume again that $\mathcal{H}$ is finite dimensional, so that the QMS is uniformly continuous. In this situation the structure of the invariant states has already been studied by several authors: Deschamps, Fagnola, Sasso and Umanità in the continuous time case [DFSU14], Baumgartner and Narnhofer [BN12], Carbone and Pautrat [CP15] for discrete time quantum channels. Our approach is very similar to the one in [DFSU14]. I emphasize that I do not get any new result here. I just believe that the simplicity of the proof makes it interesting.

The Fixed-Point algebra is necessarily atomic and as the following structure. Up to a unitary transformation[3], the Hilbert space $\mathcal{H}$ admits a decomposition as

$$\mathcal{H} = \bigoplus_{j\in J} \mathfrak{h}_j \otimes \mathfrak{m}_j,$$

where J is a finite set and such that $\mathcal{F}(\mathcal{P})$ and $\mathcal{F}(\mathcal{P})^c$ are the algebras

$$\mathcal{F}(\mathcal{P}) = \bigoplus_{j\in J} \mathcal{B}(\mathfrak{h}_j) \otimes I_{\mathfrak{m}_i}, \qquad \mathcal{F}(\mathcal{P})^c = \bigoplus_{j\in J} I_{\mathfrak{h}_j} \otimes \mathcal{B}(\mathfrak{m}_j).$$

We write P_j the orthogonal projection on $\mathfrak{h}_j \otimes \mathfrak{m}_j$. Applying Proposition I.1 we obtain the structure of the invariant states.

Theorem I.3. *Assume that $\mathcal{P}$ has a faithful normal invariant state. Then there exist unique density matrices τ_j on $\mathfrak{m}_j$ such that all the invariant states of $\mathcal{P}$ have the form:*

$$\eta = \sum_{j\in J} Tr_{\mathfrak{m}_j}[P_j \eta P_j] \otimes \tau_j, \tag{I.10}$$

It is interesting to compare this result with the one obtained in [DFSU14] that I recall later in this chapter (Theorem II.3). Indeed, the latter is based on a decomposition of the Decoherence-Free algebra $\mathcal{N}(\mathcal{P})$ instead.

Proof. Notice that the algebra $\mathcal{R}(\mathcal{F}(\mathcal{P}), \mathcal{F}(\mathcal{P})^c)$ can be identified with $\bigoplus_{j\in J} \mathcal{B}(\mathfrak{h}_j \otimes \mathfrak{m}_j)$. We denote by $E_{\mathcal{R}}$ the conditional expectation on this algebra defined by $X \mapsto \sum_{j\in J} P_j X P_j$. It is clear that $E_{\mathcal{F}} \circ E_{\mathcal{R}} = E_{\mathcal{F}}$, so that for any invariant density matrix η and any $X \in \mathcal{B}(\mathcal{H})$,

$$\mathrm{Tr}[E_{\mathcal{R}}(\eta) X] = \mathrm{Tr}[\eta E_{\mathcal{R}}(X)] = \mathrm{Tr}[\eta E_{\mathcal{F}}(X)] = \mathrm{Tr}[\eta X].$$

[3] To be more rigorous we should introduce a new Hilbert space $\mathcal{K}$ and a unitary operator from $\mathcal{H}$ to $\mathcal{K}$. However I prefer to omit this point here and I make the identification between the two Hilbert spaces.

As a consequence, we get $E_{\mathcal{R}}(\eta) = \eta$. Furthermore, notice that the spaces $\mathcal{B}(\mathfrak{h}_j \otimes \mathfrak{m}_j)$ are $\mathcal{P}$-invariant, as both $\mathcal{B}(\mathfrak{h}_j)$ and $\mathcal{B}(\mathfrak{m}_j)$ are. We denote by $\mathcal{P}^j$ the restriction of $\mathcal{P}$ to $\mathcal{B}(\mathfrak{h}_j \otimes \mathfrak{m}_j)$. A simple computation shows that $P_j \eta P_j$ is an invariant functional of $\mathcal{P}^j$:

$$\mathrm{Tr}[P_j \eta P_j \mathcal{P}_t^j(X)] = \mathrm{Tr}[P_j \eta P_j X], \qquad \forall X \in \mathcal{B}(\mathfrak{h}_j \otimes \mathfrak{m}_j),\ \forall t \geq 0.$$

Then $\mathcal{F}(\mathcal{P}_j) = \mathcal{B}(\mathfrak{h}_j) \otimes I_{\mathfrak{m}_j}$, so that it is a factor and we can apply Proposition I.1: there exists a unique density matrix τ_j such that $P_j \eta_j P_j = \mathrm{Tr}_{\mathfrak{m}}[P_j \eta P_j] \otimes \tau_j$. This concludes the proof. □

II Environment-Induced Decoherence

Even when there is no Return to Equilibrium, another more general phenomenon can occur, which is *Environment-Induced Decoherence* (EID). Recall the role played in Definition 1 by the algebra of effective observables $\mathcal{M}_0$. Under the hypothesis of the existence of faithful normal invariant state, it was proved in [Rob82] that this algebra and the Decoherence-Free algebra coincide (Proposition II.1). As a consequence, the analysis of Decoherence becomes quite close to the one of Return to Equilibrium. Indeed, Carbone, Sasso and Umanità proved that EID is implied by the existence of a conditional expectation $E_{\mathcal{N}}$ from $\mathcal{M}$ to $\mathcal{N}(P)$ compatible with the invariant state (Theorem 19 in [CSU15]).

In this section we prove that this is the case for any finite von Neumann algebra $\mathcal{M}$. This involves two steps. First, using the existence and the expression of the conditional expectation $E_{\mathcal{F}}$, we prove the existence of a particular invariant state, which is tracial on the Decoherence-Free algebra. Then, using modular theory and a result of Takesaki on conditional expectation, we obtain the desired result.

In subsection II. 1 below we recall the result of Carbone, Sasso and Umanità on the equivalence between EID and the existence of a certain conditional expectation. In subsection II. 2 we state and prove our main result on EID for finite von Neumann algebra. In subsection II. 3 we focus on the finite-dimensional case.

II. 1 The Decoherence-Free algebra

As we mentioned before, in the case where the semigroup has a faithful normal invariant state, the algebra of effective observables and the Decoherence-Free algebra coincide.

We emphasize this point in the following proposition and give a useful alternative formulation of the algebra which is due to Albeverio and Høegh-Krohn in [AHK78] (see also Proposition 18 in [CSU15]).

Proposition II.1. *Assume that $\mathcal{P}$ has a faithful normal invariant state σ. Then the maximal algebra on which $\mathcal{P}$ acts as a $*$-automorphism and the Decoherence-Free algebra coincide:*

$$\mathcal{M}_1 = \mathcal{N}(\mathcal{P})$$

Furthermore, one as

$$\mathcal{N}(\mathcal{P}) = \Big\{X \in \mathcal{M},\ \sigma(X^*X) = \sigma\left(\mathcal{P}_t(X^*)\mathcal{P}_t(X)\right), \\ \sigma(XX^*) = \sigma\left(\mathcal{P}_t(X)\mathcal{P}_t(X^*)\right) \quad \forall t \geq 0\Big\}.$$

The following theorem, which is crucial in our study of EID, is due to Carbone, Sasso and Umanità (Theroem 19 in [CSU15]). It is based on the proof of Theorem I.1 by Frigerio in [Fri78]. For sake-of-completness, we recall the proof.

Theorem II.1. *Assume that $\mathcal{P}$ has a faithful normal invariant state σ. Suppose that there exists a normal conditional expectation $E_{\mathcal{N}}$ from $\mathcal{M}$ to $\mathcal{N}(\mathcal{P})$, with non-empty kernel, which is compatible with σ, that is, such that $\sigma \circ E_{\mathcal{N}} = \sigma$. Then EID occurs.*

Proof. Assume that such a normal conditional expectation exists, that we again denote by $E_{\mathcal{N}}$. We write $\mathcal{M}_1 = \mathrm{Ker}\ \ E_{\mathcal{N}}$. By hypothesis $\mathcal{M}_1$ is a non-empty set, which furthermore is weakly$*$ closed as $E_{\mathcal{N}}$ is normal. Let prove that $w^* - \lim \mathcal{P}(X) = 0$ for all $X \in \mathcal{M}_1$.
As σ is faithful it induces a normal representation of $\mathcal{M}$ on some Hilbert space $\mathcal{H}_{\mathcal{M}}$ with cyclic and separating vector Ω. We write without loss of generality $X \mapsto X\Omega$ this representation. One has $\sigma(X) = \langle \Omega, X\Omega\rangle_{\mathcal{H}_{\mathcal{M}}}$. The QMS $\mathcal{P}$ induces a strongly continuous contraction semigroup $(P_t)_{t\geq 0}$ on $\mathcal{H}_{\mathcal{M}}$ by the formula $P_t(X\Omega) = \mathcal{P}_t(X)\Omega$, with $X \in \mathcal{M}$.

Now comes the main idea of Frigerio. Define the integrated form of the dissipation function for all $t \geq 0$ as the function

$$\begin{array}{rccc} D_t: & \mathcal{M}\times\mathcal{M} & \to & \mathcal{M} \\ & (X,Y) & \mapsto & \mathcal{P}_t(X^*Y) - \mathcal{P}_t(X^*)\mathcal{P}_t(Y) \end{array}$$

The map D_t measures the default for an $X \in \mathcal{M}$ to be in $\mathcal{N}(\mathcal{P})$. By the Kadison-Schwarz inequality, we have $D_t(X,X) \geq 0$ for all $X \in \mathcal{M}$, $t \geq 0$. Then for all $X, Y \in \mathcal{M}$, by the invariance of σ we have:

$$\begin{aligned}\lim_{t\to+\infty}\sigma(D_t(X,Y)) &= \lim_{t\to+\infty}\langle\Omega, X^*Y\Omega\rangle_{\mathcal{H}_\mathcal{M}} - \langle\mathcal{P}_t(X)\Omega, \mathcal{P}_t(Y)\Omega\rangle_{\mathcal{H}_\mathcal{M}}\\ &= \lim_{t\to+\infty}\langle X\Omega, (I-P_t^*P_t)Y\Omega\rangle_{\mathcal{H}_\mathcal{M}}\\ &= \langle X\Omega, (I-Q^2)Y\Omega\rangle_{\mathcal{H}_\mathcal{M}},\end{aligned}$$

where Q^2 is the strong limit of $P_t^*P_t$ as $t\to+\infty$, whose existence is assured by Foias and Sz.-Nagy unitary decomposition of contractions [NFBK10].

Now take $X\in\mathcal{M}_1$. As $\{\mathcal{P}_t(X)\}_{t\geq 0}$ is a bounded set it is compact for the weak$*$ topology and admits a cluster point $X_0\in\mathcal{M}_1$. We now show that $X_0\in\mathcal{N}(\mathcal{P})$, so that $X_0=0$, which will conclude the proof. Copying the previous computation, we get for all $t,s\geq 0$

$$\begin{aligned}\sigma(D_t(\mathcal{P}_s(X),\mathcal{P}_s(X))) &= \langle X\Omega, (P_s^*P_s - P_{t+s}^*P_{t+s})X\Omega\rangle_{\mathcal{H}_\mathcal{M}}\\ &= \sigma(D_{t+s}(X,X)) - \sigma(D_s(X,X))\\ &\underset{s\to+\infty}{\longrightarrow} 0.\end{aligned}$$

Therefore, using the Schwarz Inequality for the positive sesquilinear form $\sigma(D_t(\cdot,\cdot))$, we obtain that $\lim_{s\to+\infty}\sigma(D_t(\mathcal{P}_s(X),B))=0$, which proves the claim. □

Remarks II.1. In the previous proof, the central argument is the existence of the limit $s-\lim_{t\to+\infty}P_t^*P_t$. When one looks at the different proofs of EID, they all make use at some point of a decomposition of some contraction between a unitary part and a decaying part. There are two such decompositions: the unitary vs completely non-unitary decomposition of Foias and Sz.-Nagy ([ŁO03]) and the Jacobs, De Leeuw and Glicksberg decomposition ([BGKKS11] and [Hel11]). The main difficulty is to check that the orthogonal of the unitary part is indeed decaying in the long-time asymptotic. To prove this they need the additional hypothesis that the semigroup and the modular group associated to the invariant state commute.

Thus, our main contribution is to drop this assumption for finite von Neumann algebra.

II. 2 EID for finite von Neumann algebra

By a classical result of Takesaki [Tak72], the existence of $E_\mathcal{N}$ is equivalent to the invariance of the subalgebra $\mathcal{N}(\mathcal{P})$ under the action of the modular group associated to σ. Besides, a simple computation done in [CSU15] shows that this is always the case under the condition that the QMS and the modular group associated to σ commute[4].

In this subsection we prove that EID always holds whenever $\mathcal{M}$ is a finite von Neumann algebra. Equivalently, we assume that there exists a faithful normal *finite* normalized trace on $\mathcal{M}$, that we denote by Tr. Our main result is the following.

Theorem II.2. *Suppose that there exists a faithful normal invariant state σ and that $\mathcal{M}$ is finite. Then EID occurs.*

The idea of the proof is that the properties of the trace Tr are translated to the invariant state $\mathrm{Tr} \circ E_{\mathcal{F}}$ on $\mathcal{N}(\mathcal{P})$. Thus we define the state:

$$\sigma_{\mathrm{Tr}} = \mathrm{Tr} \circ E_{\mathcal{F}}. \tag{II.1}$$

This defines a proper faithful normal invariant state. Furthermore one has the following lemma.

Lemma II.1. *$\mathcal{N}(\mathcal{P})$ is in the centralizer of σ_{Tr}. More particularly, for all $X \in \mathcal{N}(\mathcal{P})$ and all $Y \in \mathcal{M}$,*

$$\sigma_{Tr}(XY) = \sigma_{Tr}(YX). \tag{II.2}$$

Proof of Lemma II.1. For all $X \in \mathcal{N}(\mathcal{P})$ and $Y \in \mathcal{M}$,

$$\begin{aligned}
\sigma_{\mathrm{Tr}}(XY) &= \mathrm{Tr} \circ E_{\mathcal{F}} \left(\mathcal{P}_t(XY)\right) \\
&= \mathrm{Tr} \circ E_{\mathcal{F}} \left(\mathcal{P}_t(X)\mathcal{P}_t(Y)\right) \\
&= \lim_{\lambda \to 0} \lambda \int_0^{+\infty} e^{-\lambda t} \mathrm{Tr}\left[\mathcal{P}_t(X)\mathcal{P}_t(Y)\right] dt \\
&= \lim_{\lambda \to 0} \lambda \int_0^{+\infty} e^{-\lambda t} \mathrm{Tr}\left[\mathcal{P}_t(Y)\mathcal{P}_t(X)\right] dt \\
&= \cdots = \sigma_{\mathrm{Tr}}(YX),
\end{aligned}$$

□

Remarks II.2. The algebra $\mathcal{N}(\mathcal{P})$ is exactly the maximal subalgebra on which this computation is possible (going from line 1 to line 2).

We can now prove the theorem.

[4] The same result was already proved in the more general case where there exists an (sub-)invariant weight (i.e. it is not any more a state) in [ŁO03].

Proof of Theorem II.2. Let $(L^2(\mathcal{M}), L^2(\mathcal{M})_+, J)$ be the standard representation of $\mathcal{M}$, where $L^2(\mathcal{M})_+$ is the cone of positive element and J is the modular conjugation. We write π the representation of $\mathcal{M}$ on $L^2(\mathcal{M})$. As σ_{Tr} is a faithful normal state there exists a cyclic and separating norm one vector $\Omega \in L^2(\mathcal{M})_+$ such that for all $X \in \mathcal{M}$,

$$\sigma_{\mathrm{Tr}}(X) = \langle \Omega, \pi(X)\Omega \rangle.$$

Consider the following map on $L^2(\mathcal{M})$:

$$S : \pi(X)\Omega \mapsto \pi(X^*)\Omega.$$

By Tomita-Takesaki modular theory (see [BR87] for instance), S is a closable anti-linear operator. We also denote by S its closure. Then its polar decomposition is given by

$$S = J\Delta^{1/2},$$

where Δ is the modular operator associated to Ω (or equivalently to σ_{Tr}). In particular, $\Delta = S^* S$.

Now, in order to prove that the modular group associated to σ_{Tr} leaves $\mathcal{N}(\mathcal{P})$ invariant, it is enough to prove that Δ leaves the closure of $\pi(\mathcal{N}(\mathcal{P}))\Omega$ invariant. Actually, using Lemma II.1, we prove that Δ is the identity operator on $\pi(\mathcal{N}(\mathcal{P}))\Omega$, which concludes the proof as Ω is cyclic for $\pi(\mathcal{N}(\mathcal{P}))$.

For simplicity we write X instead of $\pi(X)$. Then for all $X \in \mathcal{N}(\mathcal{P})$ and all $Y \in \mathcal{M}$, one has:

$$\begin{aligned}
\langle \Delta X\Omega, Y\Omega \rangle &= \langle S^* S X\Omega, Y\Omega \rangle \\
&= \overline{\langle SX\Omega, SY\Omega \rangle} \\
&= \overline{\langle X^*\Omega, Y^*\Omega \rangle} \\
&= \langle X\Omega, Y\Omega \rangle,
\end{aligned}$$

where in the last equality we use Lemma II.1. As Ω is cyclic for $\mathcal{M}$, we get the result. □

Before ending this section, I want to emphasize that one can prove in the same way that EID takes place under the following weaker assumption: there exists a faithful normal state φ, not necessarily invariant, with associated modular group $(\alpha_t)_{t\in\mathbb{R}}$ and such that:

1. (α_t) leaves $\mathcal{N}(\mathcal{P})$ invariant, i.e. $\alpha_t(\mathcal{N}(\mathcal{P})) \subset \mathcal{N}(\mathcal{P})$ for all $t \in \mathbb{R}$,

2. $\alpha_t \circ P_s = P_s \circ \alpha_t$ for all $t, s \geq 0$.

However, as I do not have example of EID in the case of a type II von Neumann algebra, I did not see the point in introducing this hypothesis.

II. 3 The case of a finite-dimensional Hilbert space

In this subsection we assume again that $\mathcal{H}$ is finite dimensional, so that the QMS is uniformly continuous. Furthermore, the Decoherence-Free algebra is necessarily atomic and has the following structure. Up to a unitary transformation, the Hilbert space $\mathcal{H}$ admits a decomposition as

$$\mathcal{H} = \bigoplus_{i\in I} \mathcal{H}_i \otimes \mathcal{K}_i, \tag{II.3}$$

where I is a finite set and such that $\mathcal{N}(\mathcal{P})$ and $\mathcal{N}(\mathcal{P})^c$ are the algebras

$$\mathcal{N}(\mathcal{P}) = \bigoplus_{i\in I} \mathcal{B}(\mathcal{H}_i) \otimes I_{\mathcal{K}_i}, \qquad \mathcal{N}(\mathcal{P})^c = \bigoplus_{i\in I} I_{\mathcal{H}_i} \otimes \mathcal{B}(\mathcal{K}_i). \tag{II.4}$$

We also know that in this case the generator of $\mathcal{P}$ is in Lindblad form [Lin76]. In [DFSU14] the authors studied the particular form of the generator imposed by the special structure of $\mathcal{N}(\mathcal{P})$ and deduced from this the structure of the invariant states. However we want to emphasize that the result does not depend on the Lindblad form of the generator so we prefer not to introduce it. We need the following notations.

As $\mathcal{P}$ acts as a $*$-automorphism on $\mathcal{N}(\mathcal{P})$, there exists a one-parameter group of unitary operators $(U_t)_{t\in\mathbb{R}}$ such that

$$\mathcal{P}_t(X) = U_t^* X U_t \qquad \forall X \in \mathcal{N}(\mathcal{P}), \forall t \geq 0. \tag{II.5}$$

Furthermore the $*$-automorphism group $t \mapsto U_t^* \cdot U_t$ leaves the type I von Neumann algebra $\mathcal{N}(\mathcal{P})$ invariant and consequently we can assume that $U_t \in \mathcal{N}(\mathcal{P})$ for all $t \in \mathbb{R}$ [BR87]. In particular, there exist selfadjoint operators $H_i \in \mathcal{B}(\mathcal{H}_i)$ such that the generator of the unitary group $(U_t)_{t\in\mathbb{R}}$ is

$$H = \sum_{i\in I} H_i \otimes I_{\mathcal{K}_i}.$$

Finally, we denote by P_i the orthogonal projection on $\mathcal{H}_i \otimes \mathcal{K}_i$.

Studying the more general case where $\mathcal{M} = \mathcal{B}(\mathcal{H})$ with possibly $\mathcal{H}$ infinite dimensional, the authors in [DFSU14] obtained the following theorem on the structure of the invariant states[5].

Theorem II.3 (Theorem 22 in [DFSU14]). *Assume that $\mathcal{P}$ has a faithful normal invariant state. Then there exist unique density matrices τ_i on $\mathcal{K}_i$ such that all the*

[5] In fact, in their article they assume that $\mathcal{N}(\mathcal{P})$ is an atomic algebra. Raffaella Carbone and Yan Pautrat noticed that this is actually always the case due to a result of Tomiyama [Tom59]: if their exists a normal conditional expectation from a type I von Neumann algebra to a commutative algebra, the latter has to be atomic.

invariant states of $\mathcal{P}$ have the form:

$$\eta = \sum_{i \in I} Tr_{\mathcal{K}_i}[P_i \eta P_i] \otimes \tau_i, \tag{II.6}$$

where the $\eta_i := Tr_{\mathcal{K}_i}[P_i \eta P_i]$ are positive matrices in $\mathcal{H}_i$ commuting with H_i.

We end this subsection with a discussion on the conditional expectation $E_{\mathcal{N}}$ in the case where $\mathcal{H}$ is finite dimensional. Its formal expression in terms of the decomposition of the space (II.3) and the different states $(\tau_i)_{i \in I}$ is given by

$$E_{\mathcal{N}}(X) = \bigoplus_{i \in I} \mathrm{Tr}_{\tau_i}[P_i X P_i] \otimes I_{\mathcal{K}_i}, \quad X \in \mathcal{B}(\mathcal{H}). \tag{II.7}$$

We see that it does not depend on the choice of the invariant state.

We denote by $\mathcal{S}(\mathcal{H})$ the set of states on the system, that is, the positive trace-class operators on $\mathcal{H}$ whose trace are equal to one. In the following we identify a density matrix $\rho \in \mathcal{S}(\mathcal{H})$ and the state $X \in \mathcal{B}(\mathcal{H}) \mapsto \mathrm{Tr}[\rho X]$ that it defines. In particular, we write $\rho(X)$ for the expectation of X in the state ρ.

Lemma II.2. *Let $\rho \in \mathcal{S}(\mathcal{H})$ be a state on $\mathcal{H}$. Define the state $\rho_{\mathcal{N}} = \rho \circ E_{\mathcal{N}}$. Then its associated density matrix on $\mathcal{H}$ is*

$$\rho_{\mathcal{N}} = \sum_{i} Tr_{\mathcal{K}_i}[P_i \rho P_i] \otimes \tau_i. \tag{II.8}$$

Furthermore its time evolution is given by

$$\mathcal{P}_{*t}(\rho_{\mathcal{N}}) = U_t \rho_{\mathcal{N}} U_t^* \text{ for all } t \geq 0, \tag{II.9}$$

where $(U_t)_{t \in \mathbb{R}}$ is the unitary group giving the evolution on $\mathcal{N}(\mathcal{P})$ as defined in Equation (II.5).

In a sense, the state $\rho_{\mathcal{N}}$ represents the knowledge we have on the state ρ when we only have access to the subsystem $\mathcal{N}(\mathcal{P})$. This remark is particularly relevant when $\mathcal{N}(\mathcal{P})$ is commutative and thought as representing the classical world. In this case $\mathcal{N}(\mathcal{P})$ can be identified with the algebra of essentially-bounded functions on some measured space: $\mathcal{N}(\mathcal{P}) \approx L^\infty(X, \mathcal{F})$. With this identification the state $\rho_{\mathcal{N}}$ (and not the density matrix) is a classical probability measure on $(X, \mathcal{F})$.

Proof. Remark that the correspondence $\rho \in \mathcal{B}_*(\mathcal{H}) \mapsto \rho \circ E_{\mathcal{N}}$ is the dual of the conditional expectation $E_{\mathcal{N}}$ for the duality between $\mathcal{B}_*(\mathcal{H})$ and $\mathcal{B}(\mathcal{H})$, so that for all $\rho \in \mathcal{S}(\mathcal{H})$ and $X \in \mathcal{B}(\mathcal{H})$, we have

$$\rho \circ E_{\mathcal{N}}(X) = \mathrm{Tr}[\rho E_{\mathcal{N}}(X)] = \mathrm{Tr}[\rho_{\mathcal{N}} X].$$

The middle term of this last equation gives

$$\begin{aligned}\mathrm{Tr}[\rho E_{\mathcal{N}}(X)] &= \mathrm{Tr}\Big[\rho\Big(\bigoplus_{i\in I}\mathrm{Tr}_{\tau_i}[P_iXP_i]\otimes I_{\mathcal{K}_i}\Big)\Big] \\ &= \bigoplus_{i\in I}\mathrm{Tr}\big[(P_i\rho P_i)(\mathrm{Tr}_{\tau_i}[P_iXP_i]\otimes I_{\mathcal{K}_i})\big] \\ &= \bigoplus_{i\in I}\mathrm{Tr}\big[\mathrm{Tr}_{\mathcal{K}_i}[P_i\rho P_i]\mathrm{Tr}_{\tau_i}[P_iXP_i]\big] \\ &= \bigoplus_{i\in I}\mathrm{Tr}\big[(\mathrm{Tr}_{\mathcal{K}_i}[P_i\rho P_i]\otimes\tau_i)(P_iXP_i)\big] \\ &= \mathrm{Tr}\Big[\Big(\bigoplus_{i\in I}\mathrm{Tr}_{\mathcal{K}_i}[P_i\rho P_i]\otimes\tau_i\Big)X\Big].\end{aligned}$$

As this is true for all $X\in\mathcal{B}(\mathcal{H})$, we obtain Equality (II.8). Then, using the fact that $E_{\mathcal{N}}$ and $\mathcal{P}_t$ commute for all $t\geq 0$, we get that for all $X\in\mathcal{B}(\mathcal{H})$:

$$\begin{aligned}\mathrm{Tr}[\rho_{\mathcal{N}}\mathcal{P}_t(X)] &= \mathrm{Tr}\big[\rho\,(U_t^*E_{\mathcal{N}}(X)U_t)\big] \\ &= \mathrm{Tr}\big[(U_t\rho U_t^*)\,E_{\mathcal{N}}(X)\big] \\ &= \mathrm{Tr}\big[(U_t\rho_{\mathcal{N}}U_t^*)\,X\big],\end{aligned}$$

which concludes the proof. □

III Speed of EID using quantum functional inequalities

The basis of the theory of functional inequalities such as Poincaré and log-Sobolev inequality in the context of quantum probability is well-developed. Starting from the definition of non-commutative L^p spaces in operator algebra theory (as defined in [AM82] for instance), Olkiewicz and Zegarlinski paved the way by introducing the main definitions and by proving the equivalence between Gross log-Sobolev inequality and hypercontractivity [OZ99]. It has then been studied by many authors [JZ15] [Zen14] [CM15] [KT13] [TPK14] [MSW15].

From now on we assume that $\mathcal{H}$ is *finite-dimensional* and that $\mathcal{P}$ has a faithful invariant state that we denote by σ.

Our goal is to study the speed of decoherence, using two different functional inequalities. The first one is a kind of Poincaré Inequality and is equivalent to the spectral gap in the reversible case. The second one is a kind of log-Sobolev Inequality, which has been considered in the classical case in [BT06] and in the quantum case in [CM15] and [KT13].

Let us give a brief intuition on these two inequalities. In the usual case, the variance and the relative entropy are good Lyapunov functionals as they are both decreasing along the semigroup and vanish on the multiples of identity. Therefore they can both be interpreted as a kind of distance to the algebra $\mathbb{C}I_{\mathcal{H}}$. However, when the semigroup is not primitive, there is no convergence toward the invariant state σ and consequently theses functionals do not vanish at infinity. Nevertheless, when EID occurs, in the long-time asymptotic $\mathcal{P}_t(X)$ is "close" to the algebra $\mathcal{N}(\mathcal{P})$. Hence, in our case, the role of the algebra $\mathbb{C}I_{\mathcal{H}}$ is replaced by the algebra $\mathcal{N}(\mathcal{P})$. In order to quantify first the distance to this algebra then the speed at which the convergence takes place, we can give alternative definition of the variance and the relative entropy that vanish only on the Decoherence-Free algebra.

In subsection III. 1 below we recall the usual definitions of the Poincaré and the log-Sobolev Inequalities. In subsection III. 2 we define the alternative Poincaré Inequality and do the same for the log-Sobolev Inequality in subsection III. 3. In subsection III. 4 we prove that the Poincaré constant we define gives a faster speed of decoherence than the log-Sobolev constant, a fact with we discuss the relevance.

III. 1 Some notations

We denote by $\mathcal{B}_{\text{sa}}(\mathcal{H})$ the set of selfadjoint operators on $\mathcal{H}$, i.e. $\mathcal{B}_{\text{sa}}(\mathcal{H}) = \{X = \mathcal{B}(\mathcal{H});\ X = X^*\}$ and by $\mathcal{B}_{\text{sa}}^{+}(\mathcal{H})$ the set of positive operators on $\mathcal{H}$, i.e. $\mathcal{B}_{\text{sa}}^{+}(\mathcal{H}) = \{X \in \mathcal{B}_{\text{sa}}(\mathcal{H});\ X > 0\}$.
The L^2 scalar product with respect to σ is defined for all $X, Y \in \mathcal{B}(\mathcal{H})$ by:

$$\langle X, Y\rangle_\sigma = \operatorname{Tr}\left[\sigma^{1/2}X^*\sigma^{1/2}Y\right].$$

It defines a norm $\|\cdot\|_{2,\sigma}$ on $\mathcal{B}(\mathcal{H})$:

$$\|X\|_{2,\sigma} = \operatorname{Tr}\left[\left|\sigma^{\frac{1}{4}}X\sigma^{\frac{1}{4}}\right|^2\right]^{\frac{1}{2}}.$$

Before focusing on the speed of decoherence, we recall the definitions of the Poincaré and one form of log-Sobolev inequality that allows to quantify the speed of convergence in the case of return to equilibrium.

We recall that $\mathcal{P}$ obeys the *Detailed Balance Condition* with respect to σ if for all $X, Y \in \mathcal{B}(\mathcal{H})$:

$$\langle X, \mathcal{L}(Y)\rangle_\sigma = \langle \mathcal{L}(X), Y\rangle_\sigma. \tag{III.1}$$

For short we will say that $\mathcal{P}$ (or $\mathcal{L}$) is reversible. The *Dirichlet form* of $\mathcal{L}$ is defined for all $X \in \mathcal{B}(\mathcal{H})$ as:

$$\mathcal{E}_\sigma(X) = -\langle X, \mathcal{L}(X)\rangle_\sigma$$

The *Variance* with respect to σ is defined for all $X \in \mathcal{B}(\mathcal{H})$ as

$$\mathrm{Var}_\sigma(X) = \|X - \sigma(X) I_{\mathcal{H}}\|^2_{2,\sigma}.$$

The *Relative Entropy* with respect to σ is defined for all $\rho \in \mathcal{S}(\mathcal{H})$ as

$$D(\rho\|\sigma) = \mathrm{Tr}\left[\rho\left(\log\rho - \log\sigma\right)\right].$$

Definition III.1 (Poincaré and Log-Sobolev Inequality:)**.**

- *$\mathcal{L}$ is said to satisfy a* Poincaré inequality *with constant $\lambda > 0$ if for all $X \in \mathcal{B}_{sa}(\mathcal{H})$:*

$$\lambda\, Var_\sigma(X) \le \mathcal{E}_\sigma(X). \tag{III.2}$$

  We denote by $\lambda(\mathcal{L})$ the optimal constant in this inequality.

- *$\mathcal{L}$ is said to satisfy a* log-Sobolev Inequality *with constant $\alpha > 0$ if for all $\rho \in \mathcal{S}(\mathcal{H})$:*

$$\alpha D(\rho\|\sigma) \le -Tr[\mathcal{L}_*(\rho)\left(\log\rho - \log\sigma\right)]. \tag{III.3}$$

  We denote by $\alpha(\mathcal{L})$ the optimal constant in the last inequality.

In the finite-dimensional case, those two constants are non-zero if and only if the semigroup is *primitive*, i.e. if it has a unique invariant state σ and it converges toward this state, that is

$$\lim_{t\to+\infty} \mathcal{P}_t(X) = \sigma(X) I_{\mathcal{H}}.$$

In this case there is a unique invariant state which consequently is used as a reference state in order to introduce the L^2 norm. In our case, the state σ_{Tr} will play the role of the reference state. Useful properties of σ_{Tr} with respect to the scalar product it defines are listed in the following lemma.

Lemma III.1. *For all $X, Y \in \mathcal{B}(\mathcal{H})$*

$$\langle E_{\mathcal{N}}(X), Y\rangle_{\sigma_{Tr}} = \langle E_{\mathcal{N}}(X), E_{\mathcal{N}}(Y)\rangle_{\sigma_{Tr}} = \langle X, E_{\mathcal{N}}(Y)\rangle_{\sigma_{Tr}}. \tag{III.4}$$

In particular, $E_{\mathcal{N}}$ is the orthogonal projection on $\mathcal{N}(\mathcal{P})$ with respect to the scalar product $\langle\cdot,\cdot\rangle_{\sigma_{Tr}}$ and Ker $E_{\mathcal{N}} = \mathcal{N}(\mathcal{P})^\perp$.

Proof. The proof relies on Lemma II.1 which, in the finite-dimensional case implies that σ_{Tr} (as a density matrix) commutes with the elements of $\mathcal{N}(\mathcal{P})$: for every $X \in \mathcal{B}(\mathcal{H})$, $[\sigma_{\mathrm{Tr}}, E_{\mathcal{N}}(X)] = 0$. Consequently, using Equation (II.7), we obtain

$$\begin{aligned}
\langle E_{\mathcal{N}}(X), Y\rangle_{\sigma_{\mathrm{Tr}}} &= \mathrm{Tr}\left[\sigma_{\mathrm{Tr}}^{1/2} E_{\mathcal{N}}(X^*)\sigma_{\mathrm{Tr}}^{1/2} Y\right] \\
&= \mathrm{Tr}\left[\sigma_{\mathrm{Tr}} E_{\mathcal{N}}(X^*) Y\right] \\
&= \sum_{i\in I} \mathrm{Tr}\Big[\sigma_{\mathrm{Tr}} \left(\mathrm{Tr}_{\mathcal{T}_i}\left[P_i X^* P_i\right] \otimes I_{\mathcal{K}_i}\right)\left(P_i Y P_i\right)\Big] \\
&= \sum_{i\in I} \mathrm{Tr}\left[\sigma_{\mathrm{Tr}}^{1/2} \left(\mathrm{Tr}_{\mathcal{T}_i}\left[P_i X^* P_i\right] \otimes I_{\mathcal{K}_i}\right)\sigma_{\mathrm{Tr}}^{1/2} \left(\mathrm{Tr}_{\mathcal{T}_i}\left[P_i Y P_i\right] \otimes I_{\mathcal{K}_i}\right)\right] \\
&= \langle E_{\mathcal{N}}(X), E_{\mathcal{N}}(Y)\rangle_{\sigma_{\mathrm{Tr}}}
\end{aligned}$$

As the last expression is symmetric in the role of X and Y, we obtain Equation (III.4). □

Remarks III.1. Looking at the proof, we see that Equation (III.4) is still true if one replaces the scalar product $\langle\cdot,\cdot\rangle_{\sigma_{\mathrm{Tr}}}$ by either the Hilbert-Schmidt scalar product $(X,Y) \mapsto \langle X, Y\rangle_{HS} = \mathrm{Tr}[X^*Y]$ or the following one: $(X,Y) \mapsto \mathrm{Tr}[\sigma_{\mathrm{Tr}} X^* Y]$. For the latter, it will even stay true for any other invariant state. This feature stems from two important points. First the fact that $\mathcal{N}(\mathcal{P})$ is a *sufficient* algebra for the set of $\mathcal{P}$-invariance states (see [HOT81] for instance), in the sense that $\sigma \circ E_{\mathcal{N}} = \sigma$. The second point is the unicity, up to unitary equivalence, of the standard representation of the von Neumann algebra $\mathcal{B}(\mathcal{H})$.

However, the fact that Equation (III.4) holds for $\langle\cdot,\cdot\rangle_{\sigma_{\mathrm{Tr}}}$ is specific to this choice of invariant state. This has to be interpreted as a special feature of the injective map $X \mapsto \sigma_{\mathrm{Tr}}^{1/4} X \sigma_{\mathrm{Tr}}^{1/4}$ from the "L^∞-space" $\mathcal{B}(\mathcal{H})$ into the "L^2-space" $(\mathcal{B}(\mathcal{H}), \langle\cdot,\cdot\rangle_{HS})$.

From now on, whenever there is no ambiguity, we will forget the subscript σ_{Tr} in the definitions of the L^2 norm and the scalar product.

III. 2 The Decoherence-Free Variance and Poincaré Inequality

Usually, the variance of an observable $X \in \mathcal{B}(\mathcal{H})$ in the state σ represents the square of the norm of $(\mathbb{C}I_{\mathcal{H}})^{\perp}$, with respect to the scalar product $\langle,\rangle_\sigma$. That is, $\mathrm{Var}_\sigma(X) = \left\|\mathrm{Proj}_{(\mathbb{C}I)^\perp}(X)\right\|_{2,\sigma}^2$. We want to keep this geometric interpretation of the variance.

By Lemma III.1, the conditional expectation $E_{\mathcal{N}}$ is the orthogonal projection on $\mathcal{N}(\mathcal{P})$ with respect to the scalar product defined by σ_{Tr}. As a consequence the following definition appears as the logical analogue of the traditional variance.

Definition III.2. *We define the* Decoherence-Free Variance *(DF-variance) for all* $X \in \mathcal{B}(\mathcal{H})$*, as the square of the norm of the projection of* X *onto the orthogonal of* $\mathcal{N}(\mathcal{P})$*, that is:*

$$Var_{\mathcal{N}}(X) = \|X - E_{\mathcal{N}}(X)\|_2^2. \tag{III.5}$$

Note that we get the usual definition when $\mathcal{N}(\mathcal{P}) = \mathbb{C}I$, so that the DF-variance is indeed a generalization of the variance. The following lemma emphasizes the particular choice of σ_{Tr} as a reference state.

Lemma III.2. *For all* $X \in \mathcal{B}_{sa}$*, one has*

$$Var_{\mathcal{N}}(X) = Var_{\sigma_{Tr}}(X) - Var_{\sigma_{Tr}}(E_{\mathcal{N}}(X)) = Var_{\sigma_{Tr}}(X) - \left[\sigma_{Tr}\left(E_{\mathcal{N}}(X)^2\right) - \sigma_{Tr}(X)^2\right]. \tag{III.6}$$

Consequently, one has $Var_{\mathcal{N}}(X) \leq Var_{\sigma_{Tr}}(X)$*.*

Proof. Remark that for all $X \in \mathcal{B}(\mathcal{H})$, by Lemma III.1, $X - E_{\mathcal{N}}(X)$ is in the orthogonal of $\mathcal{N}(\mathcal{P})$ for the concern scalar product, so that it is orthogonal to $E_{\mathcal{N}}(X) - \sigma_{\mathrm{Tr}}(X)$, which implies the first equality in Equation (III.6).
For the other other one, we use the fact that σ_{Tr} is tracial on $\mathcal{N}(\mathcal{P})$, so that $\|E_{\mathcal{N}}(X)\|_2^2 = \sigma_{\mathrm{Tr}}\left(E_{\mathcal{N}}(X)^2\right)$ and the equality follows.
The inequality between both variances is then straightforward.. □

Because of the comparison between both variances, one can expect a kind of Poincaré inequality to hold for the DF-variance even if it is not the case for the usual one.

Theorem III.1. *For all* $X \in \mathcal{B}_{sa}(\mathcal{H})$*, one has an exponential speed of EID in terms of the DF-variance:*

$$Var_{\mathcal{N}}(\mathcal{P}_t(X)) \leq e^{-2\lambda(\mathcal{L})t}\, Var_{\mathcal{N}}(X), \tag{III.7}$$

where $\lambda(\mathcal{L})$ *is the best constant* $\lambda > 0$ *such that the following* Decoherence-free Poincaré inequality *holds:*

$$\lambda\, Var_{\mathcal{N}}(X) \leq \mathcal{E}(X), \qquad X \in \mathcal{B}_{sa}(\mathcal{H}). \tag{III.8}$$

Moreover, if $\mathcal{L}$ *is reversible, then* $\lambda(\mathcal{L})$ *coincide with the spectral gap of* $\mathcal{L}$*.*

Proof. The proof is the same as for the usual Poincaré Inequality. In order to prove the first part of the Proposition we compute the derivative of $\mathrm{Var}_{\mathcal{N}}(\mathcal{P}_t(X))$. First remark that $\sigma_{\mathrm{Tr}}\left(E_{\mathcal{N}}(\mathcal{P}_t(X))^2\right) = \sigma_{\mathrm{Tr}}\left(E_{\mathcal{N}}(X)^2\right)$ and that $\sigma_{\mathrm{Tr}}(\mathcal{P}_t(X)) = \sigma_{\mathrm{Tr}}(X)$ for all $t \geq 0$. Then, using Equation (III.6), we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\mathrm{Var}_{\mathcal{N}}(\mathcal{P}_t(X)) &= \frac{\partial}{\partial t}\mathrm{Var}_{\sigma_{\mathrm{Tr}}}(\mathcal{P}_t(X)) \\ &= -2\mathcal{E}(X) \\ &\leq -2\lambda(\mathcal{L})\mathrm{Var}_{\mathcal{N}}(\mathcal{P}_t(X)).\end{aligned}$$

The result follows by integrating this inequality. Now if $\mathcal{L}$ is reversible it can be diagonalized in some orthonormal basis. We write $\lambda_1, \ldots, \lambda_n$ its eigenvalues associated to the orthogonal projections $Q_0, Q_1, \ldots, Q_n$ and such that $0 = \lambda_0 \leq -\lambda_1 \leq \cdots \leq -\lambda_n$. As proved in [CSU13], $\mathcal{N}(\mathcal{P})$ coincide with the kernel of $\mathcal{L}$, so that Q_0 is the orthogonal projection on $\mathcal{N}(\mathcal{P})$. Then for all $X \in \mathcal{B}(\mathcal{H})$

$$\begin{aligned}\mathcal{E}(X) &= \sum_{k=1}^{n} -\lambda_k |\langle Q_k, X\rangle|^2 \\ &\geq -\lambda_1 \left\| X - E_{\mathcal{N}}(X)\right\|_2^2 \\ &= -\lambda_1 \mathrm{Var}_{\mathcal{N}}(X).\end{aligned}$$

As equality is achieve for P_1, the claim is proved. □

III. 3 The Decoherence-Free Entropy and log-Sobolev Inequality

We recall that for a state $\rho \in \mathcal{S}(\mathcal{H})$, we put $\rho_{\mathcal{N}} = \rho \circ E_{\mathcal{N}}$.

Definition III.3. *We define the* Decoherence-Free relative entropy *(DF-relative entropy) for all $\rho \in \mathcal{S}(\mathcal{H})$ (view as a density matrix) as*

$$D(\,\rho \,\|\, \mathcal{N}\,) := D(\,\rho \,\|\, \rho_{\mathcal{N}}\,)\,. \tag{III.9}$$

The DF-relative entropy represents the information lost in the environment during the process of decoherence. It also reduces to the usual relative entropy in the case $\mathcal{N}(\mathcal{P}) = \mathbb{C}I$. As for Lemma III.2, the specific choice of σ_{Tr} as a reference state is motivated by the following lemma.

Lemma III.3. *For all state $\rho \in \mathcal{S}(\mathcal{H})$, one has:*

$$D(\,\rho \,\|\, \mathcal{N}\,) = D(\,\rho \,\|\, \sigma_{Tr}\,) - D(\,\rho_{\mathcal{N}} \,\|\, \sigma_{Tr}\,) \leq D(\,\rho \,\|\, \sigma_{Tr}\,)\,. \tag{III.10}$$

Consequently, $D(\,\rho \,\|\, \mathcal{N}\,) \leq D(\,\rho \,\|\, \sigma_{Tr}\,)$.

Proof. For simplicity we assume that the cardinal of I is 1 in (II.3). The proof is similar in the general case. We write $\mathcal{H} = \mathfrak{h} \otimes \mathcal{K}$, with $\dim \mathfrak{h} = N$, so that $\mathcal{N}(\mathcal{P}) = \mathcal{B}(\mathfrak{h}) \otimes I_{\mathcal{K}}$. As a consequence, for $\rho \in \mathcal{S}(\mathcal{H})$, we have $\rho_{\mathcal{N}} = \mathrm{Tr}_{\mathcal{K}}[\rho] \otimes \tau$.
Recall that if $X \in \mathcal{B}^+_{\mathrm{sa}}(\mathfrak{h})$ and $Y \in \mathcal{B}^+_{\mathrm{sa}}(\mathcal{K})$, then $\log(X \otimes Y) = (\log X) \otimes I_{\mathcal{K}} + I_{\mathcal{H}} \otimes (\log Y)$. First we compute the left-hand side of Equation (III.10):

$$\begin{aligned} D(\,\rho \,\|\, \mathcal{N}\,) &= \mathrm{Tr}\left[\rho\,(\log\rho - \log\rho_{\mathcal{N}})\right] \\ &= \mathrm{Tr}\left[\rho\log\rho\right] - \mathrm{Tr}\Big[\rho\Big(\log \mathrm{Tr}_{\mathcal{K}}[\rho] \otimes I_{\mathcal{K}} + I_{\mathcal{H}} \otimes \log\tau\Big)\Big] \\ &= \mathrm{Tr}\left[\rho\log\rho\right] - \mathrm{Tr}\Big[\mathrm{Tr}_{\mathcal{K}}[\rho]\log\mathrm{Tr}_{\mathcal{K}}[\rho]\Big] - \mathrm{Tr}\Big[\mathrm{Tr}_{\mathcal{H}}[\rho]\log\tau\Big]. \end{aligned}$$

Now we compute the first term of the right-hand side:

$$\begin{aligned} D(\,\rho \,\|\, \sigma_{\mathrm{Tr}}\,) &= \mathrm{Tr}\Big[\rho\Big(\log\rho - \log\sigma_{\mathrm{Tr}}\Big)\Big] \\ &= \mathrm{Tr}\left[\rho\log\rho\right] - \mathrm{Tr}\left[\rho\left(\log\frac{I_{\mathcal{H}}}{N} \otimes I_{\mathcal{K}} + I_{\mathcal{H}} \otimes \log\tau\right)\right] \\ &= \mathrm{Tr}\left[\rho\log\rho\right] + \log N - \mathrm{Tr}\Big[\rho\,(I_{\mathcal{H}} \otimes \log\tau)\,\Big] \\ &= D(\,\rho \,\|\, \mathcal{N}\,) + \mathrm{Tr}\Big[\mathrm{Tr}_{\mathcal{K}}[\rho]\log\mathrm{Tr}_{\mathcal{K}}[\rho]\Big] + \log N \\ &= D(\,\rho \,\|\, \mathcal{N}\,) + D\left(\,\mathrm{Tr}_{\mathcal{K}}[\rho] \,\Big\|\, \frac{I_{\mathcal{H}}}{N}\,\right) \end{aligned}$$

Finally, we have $D(\,\rho_{\mathcal{N}} \,\|\, \sigma_{\mathrm{Tr}}\,) = D\big(\,\mathrm{Tr}_{\mathcal{K}}[\rho] \otimes \tau \,\|\, \frac{I_{\mathcal{H}}}{N} \otimes \tau\,\big) = D\big(\,\mathrm{Tr}_{\mathcal{K}}[\rho] \,\|\, \frac{I_{\mathcal{H}}}{N}\,\big) + D(\,\tau \,\|\, \tau\,) = D\big(\,\mathrm{Tr}_{\mathcal{K}}[\rho] \,\|\, \frac{I_{\mathcal{H}}}{N}\,\big)$ where we used the additivity of the relative entropy. □

From this lemma we see that, as for the DF-variance, the DF-relative entropy is smaller than the relative entropy. One can now define a kind of log-Sobolev inequality which gives the exponential rate of convergence of the DF-entropy.

Theorem III.2. *Define the* Decoherence-Free log-Sobolev Inequality *as the existence of* $\alpha > 0$ *such that for all state* $\rho \in \mathcal{S}(\mathcal{H})$,

$$\alpha D(\,\rho \,\|\, \mathcal{N}\,) \leq -Tr[\mathcal{L}_*(\rho)\,(\log\rho - \log\sigma_{Tr})]. \tag{III.11}$$

Let $\alpha_{\mathcal{N}}(\mathcal{L})$ *be the best constant in the previous inequality. Then for all* $\rho \in \mathcal{S}(\mathcal{H})$,

$$D(\,\mathcal{P}_{*t}(\rho) \,\|\, \mathcal{N}\,) \leq e^{-\alpha_{\mathcal{N}}(\mathcal{L})t} D(\,\rho \,\|\, \mathcal{N}\,) \ \textit{for all } t \geq 0. \tag{III.12}$$

Proof. Writing $\rho_t = \mathcal{P}_{*t}(\rho)$ for all $t \geq 0$, we prove that:

$$\begin{aligned} \frac{\partial}{\partial t} D(\rho_t \,\|\, \mathcal{N}) &= \frac{\partial}{\partial t} D(\rho_t \,\|\, \sigma_{\mathrm{Tr}}) \\ &= \mathrm{Tr}[\mathcal{L}_*(\rho_t)\,(\log\rho_t - \log\sigma_{\mathrm{Tr}})]. \end{aligned}$$

This is equivalent to $\frac{\partial}{\partial t} D\left(\mathcal{P}_{*t}(\rho_{\mathcal{N}}) \,\|\, \sigma_{\mathrm{Tr}} \right) = 0$, that is, $D\left(\mathcal{P}_{*t}(\rho_{\mathcal{N}}) \,\|\, \sigma_{\mathrm{Tr}} \right)$ is constant in time. Indeed, the computation of $\frac{\partial}{\partial t} D\left(\rho_t \,\|\, \sigma_{\mathrm{Tr}}\right)$ is the same as for the usual case so we do not present it.

Fix a density matrix $\rho \in \mathcal{S}(\mathcal{H})$. For simplicity, we again assume that the cardinal of I is 1. The key point here is the unitary evolution of the density matrix $\rho_{\mathcal{N}}$, given by Equation (II.9) in Lemma II.2. It immediately gives

$$\mathrm{Tr}\Big[\mathcal{P}_{*t}\left(\rho_{\mathcal{N}}\right)\Big(\log \mathcal{P}_{*t}\left(\rho_{\mathcal{N}}\right)\Big)\Big] = \mathrm{Tr}\left[\rho_{\mathcal{N}} \log \rho_{\mathcal{N}}\right] \text{ for all } t \geq 0.$$

Now notice that $[\sigma_{\mathrm{Tr}}, U_t] = 0$ for all $t \geq 0$ as σ_{Tr} is an invariant state. This implies that $U_t^* \left(\log \sigma_{\mathrm{Tr}}\right) U_t = \log \sigma_{\mathrm{Tr}}$. Consequently

$$\begin{aligned}
\mathrm{Tr}\left[\mathcal{P}_{*t}\left(\rho_{\mathcal{N}}\right) \log \sigma_{\mathrm{Tr}}\right] &= \mathrm{Tr}\left[\left(U_t \rho_{\mathcal{N}} U_t^*\right) \log \sigma_{\mathrm{Tr}}\right] \\
&= \mathrm{Tr}\left[\rho_{\mathcal{N}}\left(U_t^* \log \sigma_{\mathrm{Tr}} U_t\right)\right] \\
&= \mathrm{Tr}\left[\rho_{\mathcal{N}} \log \sigma_{\mathrm{Tr}}\right] \text{ for all } t \geq 0.
\end{aligned}$$

We have proved that $D\left(\mathcal{P}_{*t}(\rho_{\mathcal{N}}) \,\|\, \sigma_{\mathrm{Tr}} \right) = D\left(\rho_{\mathcal{N}} \,\|\, \sigma_{\mathrm{Tr}} \right)$. □

III. 4 Comparison between the two constants

In this subsection we prove the following.

Theorem III.3. *Let $\mathcal{P}$ be a reversible QMS on $\mathcal{B}(\mathcal{H})$, with generator $\mathcal{L}$. Then the DF-log-Sobolev constant $\alpha_{\mathcal{N}}(\mathcal{L})$ and the spectral gap $\lambda(\mathcal{L})$ satisfy:*

$$\alpha_{\mathcal{N}}(\mathcal{L}) \leq 2\lambda(\mathcal{L}). \tag{III.13}$$

To prove this we reduce the problem to the case of a primitive QMS, for which the result was proved in [KT13] Theorem 16. Notice that our constant is twice theirs, which implies the 2 in Inequality (III.13).

Proof. We will need the structure introduced in subsection II. 3, Equations (II.3) and (II.4). Additionally to this notations, we denote by $\mathcal{P}^i$, $i \in I$, the restriction of the semigroup to $\mathcal{B}(\mathcal{K}_i)$ (which is an invariant algebra).

By definition of EID, each $\mathcal{P}_i$ is a primitive QMS with unique invariant state τ_i, which furthermore is reversible if $\mathcal{P}$ is. Denote by λ_i is spectral gap and by α_i its log-Sobolev constant. By the result mentioned above we have $\alpha_i \leq \lambda_i$, so that $\min_{i \in I} \alpha_i \leq \min_{i \in I} \lambda_i$.

Now, looking at the spectrum of $\mathcal{L}$, we have necessarily $\lambda(\mathcal{L}) = \min_{i\in I}\lambda_i$. Concerning the minimum of the α_i, it is not difficult to prove, but not needed, that it corresponds to the DF log-Sobolev constant of the restriction $\mathcal{P}^{\mathrm{da}}$ of $\mathcal{P}$ on $\mathcal{N}(\mathcal{P})'$, which is also an invariant algebra (the superscript *da* stands for Decoherence Affected, a notation introduced in [DFSU14]).

We are left to compare $\min_{i\in I}\alpha_i$ with $\alpha_{\mathcal{N}}(\mathcal{L})$. Thus, we need to prove that

$$\alpha_{\mathcal{N}}(\mathcal{L}) \le \alpha(\mathcal{L}^{\mathrm{da}}), \tag{III.14}$$

where $\mathcal{L}^{\mathrm{da}}$ is the generator of $\mathcal{P}^{\mathrm{da}}$. Again, it is enough to do this in the case where the cardinal of I is one. We put $\mathcal{H} = \mathfrak{h}\otimes\mathcal{K}$, so that $\mathcal{N}(\mathcal{P}) = \mathcal{B}(\mathfrak{h})\otimes I_{\mathcal{K}}$.
Let ρ be a state on $\mathcal{K}$. We have to prove that:

$$\alpha_{\mathcal{N}}(\mathcal{L}) D(\rho\,\|\,\tau) \le -\mathrm{Tr}\big[\mathcal{L}_*^{\mathrm{da}}(\rho)\,(\log\rho - \log\tau)\big].$$

Let η be any state on $\mathfrak{h}$. Then, as

$$D(\eta\otimes\rho\,\|\,(\eta\otimes\rho)_{\mathcal{N}}) = D(\eta\otimes\rho\,\|\,\eta\otimes\tau) = D(\rho\,\|\,\tau),$$

we have

$$\alpha_{\mathcal{N}}(\mathcal{L}) D(\rho\,\|\,\tau) \le -\mathrm{Tr}\Big[\mathcal{L}_*(\eta\otimes\rho)\Big(\log(\eta\otimes\rho) - \log(\frac{I_{\mathfrak{h}}}{N}\otimes\tau)\Big)\Big].$$

To conclude the proof, we show that the right-hand side of the two previous inequalities coincide.
Notice first that, if we denote by H the selfadjoint operator, generator of the unitary group defined by Equation (II.5), we have

$$\mathcal{L}_*(\eta\otimes\rho) = i[H,\eta]\otimes\rho + \eta\otimes\mathcal{L}_*^{\mathrm{da}}(\rho).$$

Also notice that the trace of $[H,\eta]$ multiplied by anything that commutes with η is 0, such as it is the case for $\log(\eta\otimes\rho) - \log(\frac{I_{\mathfrak{h}}}{N}\otimes\tau)$. Finally, we get

$$\begin{aligned}\mathrm{Tr}\Big[\mathcal{L}_*(\eta\otimes\rho)\Big(\log(\eta\otimes\rho) - \log(\frac{I_{\mathfrak{h}}}{N}\otimes\tau)\Big)\Big] &= \mathrm{Tr}\big[\mathcal{L}_*^{\mathrm{da}}(\rho)\,(\log\rho - \log\tau)\big] \\ &\quad + \mathrm{Tr}[\mathcal{L}_*^{d}a(\rho)]\mathrm{Tr}[\log\eta - \log\frac{I_{\mathfrak{h}}}{N}] \\ &= \mathrm{Tr}\big[\mathcal{L}_*^{\mathrm{da}}(\rho)\,(\log\rho - \log\tau)\big],\end{aligned}$$

where in the last step we used that $\mathcal{P}_*^{\mathrm{da}}$ is trace-preserving. □

As was discussed in the introduction of this manuscript (Section V) the main interest of the Decoherence-Free log-Sobolev Inequality would be to take into account entanglement effects. As such this definition would be even more relevant if one could prove that Inequality (III.14) is strict. More precisely, looking at the proof of Theorem III.3, what we want to compare are the two constants $\alpha_{\mathcal{N}}(\mathcal{L})$ and $\alpha(\mathcal{L}^{\mathrm{da}})$. Remark that in particular equality is achieved when the minimum is taken over separable states, which shows that strict inequality is indeed a manifestation of the quantum world.

IV Conclusion and perspectives

As I mentioned at the beginning of this chapter, the study presented here is still a work-in-progress.

Concerning EID for finite von Neumann algebra, it would be necessary to obtain physically relevant examples of QMS with non-trivial Decoherence-Free algebra. Some examples were already proposed in the case where $\mathcal{N}(\mathcal{P})$ is commutative in the original article of Blanchard and Olkiewicz [BO03]. The same holds for the recipe to study the structure of the invariant states. A good candidate would be the quantum Markov semigroup defined by Majewski, Olkiewicz and Zegarliński in [MOZ98] in the case of quantum spin systems on a lattice.

Concerning the speed of decoherence, a lot has yet to be understood. Let me list a few questions:

- Looking at the central role played by the invariant state σ_{Tr}, which was defined in the more general case of a finite von Neumann algebra, it seems that the definitions of the DF-variance and the DF-entropy can be extended to this setting. In this case it is not always true that the semigroup has a spectral gap, so these definitions make even more sense.

- In a recent article [MHFW16], Müller-Hermes, Stilck França and Wolf computed the optimal log-Sobolev constant for the depolarizing channel presented in the introduction of this manuscript. Can we do the same for the DF log-Sobolev constant of some quantum channel on a bipartite system? In this way we could compare explicitly the two constants and see how much they differ. I emphasize once again on the particular relevance of their difference.

- One way to obtain in general a lower-bound on the DF log-Sobolev constant would be to study Hypercontractivity Property and its associated log-Sobolev Inequality. Indeed ,the log-Sobolev Inequality we generalized to the EID setting was introduced only recently in [BT06] and is named the 1-log Sobolev Inequality in the literature ([TPK14] [CM15] [MSW15]). The usual log-Sobolev Inequality gives a constant $\alpha_2(\mathcal{L})$ which was proved, under some regularity assumptions, to be smaller than $\alpha(\mathcal{L})$. In general, it is easier to obtain lower bound on α_2 using hypercontractivity [MSW15].

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Émergence de dynamiques classiques en probabilités quantiques

Résumé : Cette thèse se consacre à l'étude de certaines passerelles existantes entre les probabilités dîtes classiques et la théorie des systèmes quantiques ouverts.

Le but de la première partie de ce manuscrit est d'étudier l'émergence de bruits classiques dans l'équation de Langevin quantique. Cette équation sert à modéliser l'action d'un bain quantique sur un petit système dans l'approximation markovienne. L'analogue en temps discret de cette équation est décrit par le schéma des interactions quantiques répétées étudier par Stéphane Attal et Yan Pautrat. Dans des travaux antérieurs, Attal et ses collaborateurs montrent que les bruits classiques naturels apparaissant dans ce cadre sont les *variables aléatoires obtuses*, dont ils étudient la structure. Mais sont-ils les seuls bruits classiques pouvant émerger, et quand est-il dans le cas général ? De même, en temps continu, il était plus ou moins admis que les seuls bruits classiques apparaissant dans l'équation de Langevin quantique sont les processus de Poisson et le mouvement brownien. Ma contribution dans ce manuscrit consiste à définir une algèbre de von Neumann pertinente sur l'environnement, dite *algèbre du bruit*, qui encode la structure du bruit. Elle est commutative si et seulement si le bruit est classique ; dans ce cas on confirme les hypothèses précédentes sur sa nature. Dans le cas général, elle permet de montrer une décomposition de l'environnement entre une partie classique maximale et une partie purement quantique.

Dans la deuxième partie, nous nous consacrons à l'étude de processus stochastiques classiques apparaissant au sein du système. La dynamique du système est quantique, mais il existe une observable dont l'évolution est classique. Cela se fait naturellement lorsque le semi-groupe de Markov quantique laisse invariante une sous-algèbre de von Neumann commutative et maximale. Nous développons une méthode pour générer de tels semi-groupes, en nous appuyons sur une définition de Stéphane Attal de certaines dilatations d'opérateurs de Markov classiques. Nous montrons ainsi que les *processus de Lévy* sur $\mathbb{R}^n$ admettent des extensions quantiques.

Nous étudions ensuite une classe de processus classiques liés aux *marches quantiques ouvertes*. De tels processus apparaissent lorsque cette fois l'algèbre invariante est le produit tensoriel de deux algèbres, l'une non-commutative et l'autre commutative. Par conséquent, bien que comportant l'aspect trajectoriel propre au processus classiques, de telles marches aléatoires sont hautement quantiques. Nous présentons dans ce cadre une approche variationnelle du *problème de Dirichlet*.

Finalement, la dernière partie est dédiée à l'étude d'un processus physique appelé *décohérence induite par l'environnement*. Cette notion est fondamentale, puisqu'elle apporte une explication dynamique à l'absence, dans notre vie de tous les jours, de phénomènes quantiques. Nous montrons qu'une telle décohérence a toujours lieu pour des systèmes ouverts décrits par des algèbres de von Neumann finies. Nous initions ensuite une étude innovante sur la vitesse de décohérence, basée sur des inégalités fonctionnelles non-commutatives, qui permet de mettre en avant le rôle de l'intrication quantique dans la décohérence.

Mots clefs : probabilités classiques et quantiques, calcul stochastique quantique, algèbres d'opérateur, extension quantique, décohérence induite par l'environnement, inégalités fonctionelles quantiques

Image en couverture : Chat de Schrödinger en intéractions répétées avec des chats classiques (dessin de Alice Merle).

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